Linear v. Nonlinear Differential Equations

The definition of a linear differential equation given in the text on p. 21 is a differential equation of the form

\[ F(t, y, y', \ldots, y^{(n)}) = 0 \]

where \( F \) is a linear function of the variables \( y^{(i)} \) with any function of just \( t \) allowed to be a coefficient. A way of thinking of this is that the equation can be written as a linear combination of the form

\[ \sum_{i=1}^{n} a_i(t)y^{(i)} = g(t) \]

meaning, for instance, that none of the \( n^{th} \) derivatives of \( y \) may be multiplied by each other or raised to a power - only linear operations of these are allowable.

Examples (linear):

1. \( 4t^2e^t + 3ty - y'' = 0 \)
2. \( y'''e^t - y = y' + 1 \)
3. \( 16y + e^ty'' - \sin(t)y'' + y'' = 10, \) because it’s equivalent to \( 16y + (e^t - \sin(t))y'' - 10 = 0 \)
4. \( \frac{1}{y'} = 6t^3 - 5, \) equivalent to \( 1 = 6t^3y' - 5y', \) or \( 6t^3y' - 5y' - 1 = 0 \)
5. \( e^{y'} - 4t = 0, \) equivalent to \( y' = \ln(4t). \)

Examples (non-linear):

1. \( y' = y^2, \) because \( y \) is being squared, which is not a linear operation
2. \( y' \cdot y'' = 4t, \) because two derivatives are being multiplied together
3. \( \frac{1}{y} - \frac{1}{y'} = 1, \) because there is no way to get the \( y \)-expressions out of the denominator without ending up with two of them multiplied together.
4. \( y' = e^y, \) because to release \( y \) from the exponent, you would have to take the log of both sides.