Solutions

1. Calculate [20% each]:
   
   (a) \[ \int_0^{\pi/6} \cos^2(t) \sin^2(t) \, dt \]
   
   \[= \int_0^{\pi/6} \left( \frac{1 + \cos(2t)}{2} \right) \left( \frac{1 - \cos(2t)}{2} \right) \, dt = \frac{1}{4} \int_0^{\pi/6} 1 - \cos^2(2t) \, dt \]
   
   \[= \frac{1}{4} \int_0^{\pi/6} 1 - \left( \frac{1 + \cos(4t)}{2} \right) \left( \frac{1 - \cos(4t)}{2} \right) \, dt = \frac{1}{8} \int_0^{\pi/6} 1 - \cos(4t) \, dt \]
   
   \[= \frac{1}{8} \left( t - \frac{1}{4} \sin(4t) \right) \bigg|_0^{\pi/6} = \frac{\pi}{48} - \frac{\sqrt{3}}{64} \]

   (b) \[ \int x \sqrt{e^x - 1} \, dx \]

   \[= \int xe^{\frac{1}{2}(x-1)} \, dx \]

   \[u = x \quad dv = e^{\frac{1}{2}(x-1)} \, dx \]

   \[du = dx \quad 2e^{\frac{1}{2}(x-1)} \]

   Then the integral becomes

   \[2xe^{\frac{1}{2}(x-1)} - 2 \int e^{\frac{1}{2}(x-1)} \, dx = 2xe^{\frac{1}{2}(x-1)} - 4e^{\frac{1}{2}(x-1)} + C = e^{\frac{1}{2}(x-1)}(2x - 4) + C \]

   (c) \[ \int \tan(\theta) \cdot (\ln(\sec(\theta)))^4 \, d\theta \]

   \[u = \ln(\sec(\theta)) \quad du = \tan(\theta) \, d\theta \]

   The integral becomes

   \[\int u^4 \, du = \frac{1}{5} u^5 + C = \frac{1}{5} (\ln(\sec(\theta)))^5 + C \]

2. Use the method of cylindrical shells to find the volume of the solid of revolution obtained by revolving about the x-axis the curve \( y = \sqrt[4]{x-1} \) from the point (1, 0) to (17, 2). [20%]

   Careful reading of this problem statement is very important: because you were required to use the method of cylindrical shells, and because the solid is created by rotating about the x-axis, the volume must be calculated as an integral with respect to y. So, even if you had the correct formula: \( V = 2\pi \int_a^b rh \), it will only correctly calculate the volume if r and h are each functions with y as the independent variable. To find the correct height and radius functions, we first solve the curve’s equation for x to get \( x = y^4 + 1 \). The height of each cylinder is not simply \( y^4 + 1 \), though: each of the cylinders extends from the place where it touches the graph on the left to the rightmost edge of the cylinder. This distance is \( h(y) = 17 - (y^4 + 1) = 16 - y^4 \). The radius of each cylinder is simply y.

   \[ V = 2\pi \int_0^2 y (16 - y^4) \, dy = \frac{128\pi}{3} \]

3. The growth rate of a population of chipmunks is proportional to the size of the population at any given time. The initial cluster of chipmunks numbers 200, and after two years the population has increased by 25%. [10% each]

   (a) Write an initial value problem that models the data in the description.

   Remember that the initial value problem refers to the differential equation together with the initial conditions, so all you needed to write for this response is:

   \[ P' = kP, \quad P(0) = 200 \]
(b) Solve the initial value problem, and use your solution to determine how many chipmunks there will be after 15 years.

Solving the equation yields \( P(t) = P_0e^{kt} = 200e^{kt} \), and to find \( k \), we use the last part of the problem. If the population increases by 25% over two years, that means \( P(2) = 250 \). Solving \( P(2) = 250 \) yields \( k = \frac{1}{2} \ln \frac{5}{4} \). Thus:

\[
P(15) = 200e^{15 \cdot \frac{1}{2} \ln \frac{5}{4}} \approx 1066 \text{ chipmunks}
\]