1. Calculate [32 pts]:

(a) \[ \int_{1}^{\ln 5} (t + 1)e^t \, dt \]

This requires integration by parts.

\[ u = t + 1 \quad dv = e^t \, dt \]
\[ du = dt \quad v = e^t \]

The integral becomes

\[ (t + 1)e^t − \int e^t \, dt \]
\[ (t + 1)e^t − e^t \bigg|_{1}^{\ln 5} = te^t \bigg|_{1}^{\ln 5} \]
\[ = 5 \ln 5 − e = \ln 25 − e \]

(b) \[ \int \cos^2(\theta) \sin^5(\theta) \, d\theta \]

This integral has an odd power of the sine function, so the method will be to rewrite the integral in terms of cosine, and leave a single power of sine on the outside to pair up with \( d\theta \).

\[ \int \cos^2(\theta) \sin^5(\theta) \, d\theta = \int \cos^2(\theta) \sin^4(\theta) \sin(\theta) \, d\theta = \int \cos^2(\theta) \left(1 − \cos^2(\theta)\right)^2 \sin(\theta) \, d\theta \]

\[ u = \cos(\theta); \quad \frac{du}{d\theta} = −\sin(\theta), \text{ so the integral becomes} \]
\[ −\int u^2 (1 − u^2)^2 \, du = −\int u^2 (1 − 2u^2 + u^4) \, du = −\int u^2 − 2u^4 + u^6 \, du \]
\[ = −\frac{1}{3}u^3 + \frac{2}{5}u^5 − \frac{1}{7}u^7 + C = −\frac{1}{3}\cos(\theta) + \frac{2}{5}\cos^3(\theta) − \frac{1}{7}\cos^7(\theta) + C \]

(c) \[ \int \frac{x^3 − \frac{1}{4}\sin(x)}{3\sqrt{x^4 + \cos(x)}} \, dx \]

\[ u = x^4 + \cos(x) \]
\[ du = (4x^3 − \sin(x))\,dx \]
\[ \frac{1}{4}du = (x^3 − \frac{1}{4}\sin(x))\,dx \]

\[ \frac{1}{4} \int \frac{du}{3\sqrt{u}} = \frac{1}{12} \int u^{-1/2} \, du = \frac{1}{6}\sqrt{u} + C = \frac{1}{6}\sqrt{x^4 + \cos(x)} + C \]

(d) \[ \int_{1}^{e^3} x^5 \ln(x) \, dx \]

\[ u = \ln(x) \quad dv = x^5 \, dx \]
\[ du = \frac{dx}{x} \quad v = \frac{x^6}{6} \]

\[ \frac{1}{6}x^6 \ln(x) − \int \frac{x^6}{6x} \, dx = \frac{1}{6}x^6 \ln(x) − \int \frac{1}{6}x^5 \, dx = \frac{1}{6}x^6 \ln(x) − \frac{1}{36}x^6 \bigg|_{1}^{e^3} \]
\[ = \frac{1}{6}e^{18} \ln(e^3) − \frac{1}{36}e^{18} − \frac{1}{6}\ln(1) + \frac{1}{36} \]
\[ = e^{18} \left( \frac{3}{6} − \frac{1}{36} \right) + \frac{1}{36} = \frac{1}{36} (17e^{18} + 1) \]
2. Find the volume of the solid of revolution generated by revolving about the x-axis the area enclosed by the functions \( y = x \) and \( y = \frac{1}{4}x^2 \). [12 pts]:

First, sketch the shape and determine the points of intersection to find that the area extends from \( x = 0 \) to \( x = 4 \), and that \( y = x \) is on top. Then use the method of washers, \( \pi \int_a^b R^2 - r^2 \):

\[
V = \pi \int_0^4 x^2 - \left( \frac{1}{4}x^2 \right)^2 \, dx = \pi \int_0^4 x^2 - \frac{1}{16}x^4 \, dx
\]

\[
= \pi \left( \frac{1}{3}x^3 - \frac{1}{80}x^5 \right) \bigg|_0^4 = \pi \left( \frac{64}{3} - \frac{1024}{80} \right)
\]

Remember that you can leave the answer in the form above if you like; if you felt unsatisfied without fully simplifying your answer, then you could have proceeded to get \( \frac{128\pi}{15} \).

3. Calculate the area of the surface of revolution generated by revolving about the x-axis the section of the curve \( f(x) = \sqrt{4 - 2x} \) extending from the point \((0, 2)\) to the point \((2, 0)\). [12 pts].

This question was made much easier by calculating \( f'(x) \) correctly - mistakes, most notably forgetting to use the chain rule, tended to make the algebra much more difficult.

\[
f'(x) = \frac{-2}{2\sqrt{4-2x}} = \frac{-1}{\sqrt{4-2x}}
\]

Thus, the integral for the surface area is:

\[
A = 2\pi \int_0^2 \sqrt{4-2x} \sqrt{\frac{1}{4-2x} + 1} \, dx = 2\pi \int_0^2 \sqrt{4-2x} \sqrt{\frac{1}{4-2x} + \frac{4-2x}{4-2x}} \, dx
\]

\[
= 2\pi \int_0^2 \sqrt{4-2x} \sqrt{\frac{5-2x}{4-2x}} \, dx = 2\pi \int_0^2 \sqrt{\frac{5-2x}{4-2x}} \, dx
\]

This integral is much easier to handle than any other form you could possibly have found for it.

\[
= 2\pi \left( \frac{1}{2} \cdot \frac{2\sqrt{3}}{2} \cdot (5-2x)^{3/2} \right) \bigg|_0^2 = -\frac{2}{3} \pi \left( 1 - (5^{3/2}) \right) = \frac{2}{3} \pi \left( 5^{3/2} - 1 \right)
\]

4. A large bag of water is leaking slowly through a tiny hole in the bottom. The water pressure surrounding the hole decreases as the bag loses water; as a consequence, the rate at which the water leaks from the bag is proportional to the mass of the water in the bag, satisfying the differential equation

\[
M'(t) = -kM(t)
\]

where \( M(t) \) is the mass of the water in the bag (in kilograms) after \( t \) minutes have elapsed.

(a) Solve the differential equation in terms of \( k \) and initial water mass \( M_0 \). [8 pts]

This equation has been solved several times in class and in the text, so we’ll skip to the punchline:

\[
M(t) = M_0 e^{-kt}
\]

(b) Assume the bag initially contains 2 kg of water, and loses 10% of its weight every minute.

i. Solve for \( k \). [5 pts]

This requires you to use the given information as a set of initial conditions: \( M_0 = 2 \), and \( M(1) = 1.8 \).

\[
1.8 = 2e^{-k}
\]

\[
0.9 = e^{-k}
\]

\[
k = -\ln(0.9) = \ln \left( \frac{10}{9} \right)
\]
ii. Use this to determine the time elapsed in minutes before the mass of the water in the bag is
reduced to 0.1 kg. [5 pts]

\[
0.1 = 2e^{-\ln(10/9)t} \\
\frac{1}{20} = e^{-\ln(10/9)t} \\
\ln \left(\frac{1}{20}\right) = -\ln \left(\frac{10}{9}\right) t
\]

\[
t = \frac{\ln(1/20)}{\ln(9/10)}
\]

which, if you want an estimate in minutes, is approximately 28.43.

5. Solve the initial value problem; write your answer as a function of \(t\). [10 pts]

\[
\frac{dy}{dt} = \frac{\sin(t)}{\sqrt{y}}, \quad y(0) = 0
\]

This and the next problem each require separation of variables, before integrating both sides.

\[
\sqrt{y} \frac{dy}{dt} = \sin(t) \, dt
\]

\[
\int \sqrt{y} \, dy = \int \sin(t) \, dt
\]

\[
\frac{2}{3} y^{3/2} = -\cos(t) + C
\]

\[
y = \left( -\frac{3}{2} \cos(t) + C \right)^{2/3}
\]

For this problem, we need now to solve for \(C\) using the initial conditions provided.

\[
0 = \left( -\frac{3}{2} + C \right)^{2/3}
\]

\[
C = \frac{3}{2}
\]

\[
y = \left( -\frac{3}{2} \cos(t) + \frac{3}{2} \right)^{2/3}
\]

6. Solve the differential equation; you may leave your answer in implicit form. [8 pts]

\[
\frac{dy}{dx} \left(\ln(y)\right) = x^2 y + 3xy
\]

\[
\frac{1}{y} \ln(y) \frac{dy}{dx} = x^2 + 3x
\]

\[
\int \frac{1}{y} \ln(y) \, dy = \int x^2 + 3x \, dx
\]

The integral on the left is a simple substitution with \(u = \ln(y)\), whence \(du = \frac{1}{y} \, dy\):

\[
\frac{1}{2} (\ln(y))^2 = \frac{1}{3} x^3 + \frac{3}{2} x^2 + C
\]

If you ignored the offer to leave this the solution in implicit form, then solving for \(y\) would have gotten
you to

\[
y = e^{\sqrt{\frac{3}{2}x^3 + 3x^2 + C}}
\]
7. A child is making a snowball in the fresh Chicago snow, and rolling it across a field. The force required to push the snowball increases linearly as the snowball increases in mass, which occurs at a rate commensurate with the cube of the square root of the distance that the snowball travels. In other words, the force, in pounds, needed to push the snowball after it has traveled $x$ feet is $\alpha x^{3/2}$ for some constant $\alpha$. Calculate, in terms of $\alpha$, the amount of work that is done by moving the snowball 25 feet. [8 pts]

This problem, just like the one on quiz 2, was full of way too much information - the only thing you needed was the force function, which was given to you explicitly! So, all you had to do was use this function in the work calculation integral.

$$\int_{0}^{25} \alpha x^{3/2} \, dx$$

$$= \frac{2}{5} \alpha x^{5/2} \bigg|_0^5 = \frac{2}{5} \alpha \cdot 25^{5/2} = 1250 \alpha \text{ ft-lbs.}$$

Bonus. Compute, using a substitution prior to integration by parts. [5 pts].

$$\int e^{2x} \ln \left(1 + e^{2x}\right) \, dx$$

$u = 1 + e^{2x}$
$du = 2e^{2x} \, dx$

Thus, the integral becomes

$$\frac{1}{2} \int \ln(u) \, du$$

$$= \frac{1}{2} (u \ln(u) - u) + C = \frac{1}{2} \left(1 + e^{2x}\right) \left(\ln \left(1 + e^{2x}\right) - 1\right) + C$$