1. \[
\int \frac{x+1}{x^4-2x^3+x^2} \, dx
\]

The denominator factors as \(x^2(x-1)^2\), so our partial fractions setup is:

\[
\frac{x+1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}
\]

Finding common denominators, we get

\[
x + 1 = A(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2
\]

This leaves us the equations

\[
B = 1; \quad A - 2B = 1; \quad A + C = 0; \quad -2A + B - C + D = 0
\]

which can be solved easily if done in that order to yield \((A, B, C, D) = (3, 1, -3, 2)\). Thus, our decomposed integral is

\[
\int \frac{3}{x} + \frac{1}{x^2} - \frac{3}{x-1} + \frac{2}{(x-1)^2} \, dx
\]

2. \[
\int \frac{x^3 - x^2 + x + 1}{x^2 + 5} \, dx
\]

Before solving this equation, we must do long division to get a proper fraction.

\[
\frac{x^3 - x^2 + x + 1}{x^2 + 5} = \frac{x^3 - 5x - x^2 - 5 + x + 1}{x^2 + 5}
\]

\[
x(x^2 + 5) - (x^2 + 5) - 4x + 6 = x - 1 + \frac{-4x + 6}{x^2 + 5}
\]

Since the denominator is irreducible, we don’t have to decompose it; we can find the integral by breaking up the sum in the numerator:

\[
\int x - 1 - \frac{4x}{x^2 + 5} + \frac{6}{x^2 + 5} \, dx = \frac{x^2}{2} - x - 2\ln(x^2 + 5) + \frac{6}{\sqrt{5}} \arctan \left( \frac{x}{\sqrt{5}} \right) + C
\]

3. \[
\int \frac{(x-1)(x+3)}{(x-2)(x+7)} \, dx
\]

In order to understand what is going on in this integral, it is perhaps easiest to expand the two factored polynomials, followed by decomposing the fractions:

\[
\frac{(x-1)(x+3)}{(x-2)(x+7)} = \frac{x^2 + 2x - 3}{x^2 + 5x - 14} = \frac{x^2 + 5x - 14 + 2x - 3 - 5x + 14}{x^2 + 5x - 14} = 1 + \frac{-3x + 11}{x^2 + 5x - 14}
\]

Now, we must decompose the fraction

\[
\frac{-3x + 11}{(x-2)(x+7)} = \frac{A}{x-2} + \frac{B}{x+7}. \quad \text{Thus,}
\]

1
\[-3x + 11 = Ax + 7A + Bx - 2B = (A + B)x + 7A - 2B\]

Solve \(A + B = -3\) and \(7A - 2B = 11\) to get \(A = \frac{5}{9}\) and \(B = -\frac{32}{9}\). Thus the antiderivative (not forgetting about the 1 that we got from separating the fraction) is:

\[x + \frac{5}{9} \ln |x - 2| - \frac{32}{9} \ln |x + 7| + C\]

4. \(\int \frac{\sin x \cos x}{(\sin x - 1)(\sin x + 3)} \, dx\)

This problem requires a substitution prior to integration by partial fractions. We let \(u = \sin(x)\), in which case \(du = \cos(x)\), and we get:

\[\int \frac{1}{(u - 1)(u + 3)} \, du\]

This is a straightforward partial fraction decomposition, resulting in the integral

\[= \frac{1}{4} \left( \ln |u - 1| + 3 \ln |u + 3| \right) + C = \frac{1}{4} \left( \ln |\sin x - 1| + 3 \ln |\sin x + 3| \right) + C\]