1. $\displaystyle\int \frac{x+1}{x^4-2x^3+x^2}\,dx$

The denominator factors as $x^2(x-1)^2$, so our partial fractions setup is:

$$\frac{x+1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Finding common denominators, we get

$$\begin{aligned} x+1 &= A(x)(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 \\ &= A(x^3-2x^2+x) + B(x^2-2x+1) + C(x^3-x^2) + Dx^2 \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B)x + B \end{aligned}$$

This leaves us the equations

$B = 1; \quad A - 2B{=}1; \quad A + C = 0; \quad -2A + B - C + D = 0$
which can be solved easily if done in that order to yield $(A, B, C, D) = (3, 1, -3, 2)$. Thus, our decomposed integral is

$$\int \frac{3}{x} + \frac{1}{x^2} - \frac{3}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$$= 3\ln|x| - \frac{1}{x} - 3\ln|x-1| - \frac{2}{x-1} + C$$

2. $\displaystyle\int \frac{x^3-x^2+x+1}{x^2+5}\,dx$

Before solving this equation, we must do long division to get a proper fraction.

$$\frac{x^3-x^2+x+1}{x^2+5} = \frac{x^3+5x-x^2-5+x+1-5x+5}{x^2+5}$$

$$= \frac{x(x^2+5)-(x^2+5)-4x+6}{x^2+5} = x - 1 + \frac{-4x+6}{x^2+5}$$

Since the denominator is irreducible, we don't have to decompose it; we can find the integral by breaking up the sum in the numerator:

$$\int x - 1 - \frac{4x}{x^2+5} + \frac{6}{x^2+5}\,dx = \frac{x^2}{2} - x - 2\ln(x^2+5) + \frac{6}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

3. $\displaystyle\int \frac{(x-1)(x+3)}{(x-2)(x+7)}\,dx$

In order to understand what is going on in this integral, it is perhaps easiest to expand the two factored polynomials, followed by decomposing the fractions:

$$\frac{(x-1)(x+3)}{(x-2)(x+7)} = \frac{x^2+2x-3}{x^2+5x-14} = \frac{x^2+5x-14+2x-3-5x+14}{x^2+5x-14} = 1 + \frac{-3x+11}{x^2+5x-14}$$

Now, we must decompose the fraction

$$\frac{-3x+11}{(x-2)(x+7)} = \frac{A}{x-2} + \frac{B}{x+7}. \text{ Thus,}$$

$$-3x + 11 = Ax + 7A + Bx - 2B = (A + B)x + 7A - 2B$$

Solve $A + B = -3$ *and* $7A - 2B = 11$ *to get* $A = \frac{5}{9}$ *and* $B = \frac{-32}{9}$. *Thus the antiderivative (not forgetting about the* 1 *that we got from separating the fraction) is:*

$$x + \frac{5}{9}\ln|x-2| - \frac{32}{9}\ln|x+7| + C$$

4. $\displaystyle\int \frac{\sin x \cos x}{(\sin x - 1)(\sin x + 3)}\,dx$

This problem requires a substitution prior to integration by partial fractions. We let $u = \sin(x)$, *in which case* $du = \cos(x)$, *and we get:*

$$\int \frac{1}{(u-1)(u+3)}\,du$$

This is a straightforward partial fraction decomposition, resulting in the integral

$$\frac{1}{4}\int \frac{1}{u-1} + \frac{3}{u+3}\,du$$

$$= \frac{1}{4}\left(\ln|u-1| + 3\ln|u+3|\right) + C = \frac{1}{4}\left(\ln|\sin x - 1| + 3\ln|\sin x + 3|\right) + C$$