1. Integrate (give your answer in terms of $t$): \[ \int \frac{1}{\sqrt{9-t^2}} \, dt \]

We should rewrite this one prior to substituting:

\[ \int \frac{1}{\sqrt{9-t^2}} \, dt = \int \frac{1}{\sqrt{9-(t/3)^2}} \, dt = \int \frac{1}{3\sqrt{1-(t/3)^2}} \, dt \]

Now, let $\frac{t}{3} = \sin \theta$, so $dt = 3 \cos \theta \, d\theta$, and, for back-substitution purposes, we have $\theta = \arcsin \left( \frac{t}{3} \right)$:

\[ \frac{1}{3} \int \frac{3 \cos \theta \, d\theta}{\sqrt{1-\sin^2 \theta}} = \int \frac{\cos \theta}{\sqrt{\cos^2 \theta}} \, d\theta = \int 1 \, d\theta \]

\[ = \theta + C = \arcsin \left( \frac{t}{3} \right) + C \]

2. Integrate (give your answer in terms of $x$): \[ \int \frac{3x}{\sqrt{x^2-4}} \, dx \]

Notice that this one could easily be done with a regular substitution! But, if you didn’t catch that, then you could have proceeded with a trigonometric substitution, as follows. First, it must be rewritten a bit:

\[ \int \frac{3x}{\sqrt{x^2-4}} \, dx = 3 \int \frac{x}{\sqrt{4 \left( \frac{x^2}{4} - 1 \right)}} \, dx = 3 \int \frac{x}{2 \sqrt{\left( \frac{x}{2} \right)^2 - 1}} \, dx = \frac{3}{2} \int \frac{x}{\sqrt{\left( \frac{x}{2} \right)^2 - 1}} \, dx \]

Now it is clear that the substitution should be $\frac{x}{2} = \sec \theta$, whence $x = 2 \sec \theta$, $dx = 2 \sec \theta \tan \theta \, d\theta$, and for back-substitution purposes, we note that $\theta = \arccsc \left( \frac{x}{2} \right)$. Rewrite the integral to get:

\[ \frac{3}{2} \int \frac{2 \sec \theta}{\sqrt{\sec^2 \theta - 1}} \cdot 2 \sec \theta \tan \theta \, d\theta = 6 \int \frac{\sec^2 \theta \tan \theta}{\sqrt{\tan^2 \theta}} \, d\theta = 6 \int \sec^2 \theta \, d\theta \]

\[ = 6 \tan \theta + C = 6 \tan \arccsc \frac{x}{2} + C \]

Again, we use a triangle to recover $\tan \arccsc \frac{x}{2}$:

From this we calculate $\tan \arccsc \frac{x}{2} = \frac{1}{2} \sqrt{x^2-4}$. Thus, the final answer becomes $3 \sqrt{x^2-4} + C$.

3. Integrate (give your answer in terms of $x$): \[ \int \frac{x^2}{(1+x^2)^{3/2}} \, dx \]

Using trigonometric substitution, let $x = \tan \theta$. Recall for our later back substitution that this means $\theta = \arctan x$. Then $dx = \sec^2 \theta \, d\theta$, and the integral becomes:

\[ \int \frac{\tan^2 \theta \sec^2 \theta}{(1+\tan^2 \theta)^{3/2}} \, d\theta = \int \frac{\tan^2 \theta \sec^2 \theta}{(\sec^2 \theta)^{3/2}} \, d\theta = \int \frac{\tan^2 \theta \sec^2 \theta}{\sec^3 \theta} \, d\theta = \int \frac{\tan^2 \theta}{\sec \theta} \, d\theta \]
\[
\int \frac{\sin^2 \theta}{\cos \theta} d\theta = \int \frac{\cos \theta}{\sin^2 \theta} d\theta
\]

\[
= \int \frac{\sin^2 \theta}{\cos \theta} d\theta = \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta = \int \frac{1}{\cos \theta} - \cos \theta d\theta = \int \sec \theta - \cos \theta d\theta
\]

\[
= \ln |\sec \theta + \tan \theta| - \sin \theta + C = \ln |\sec \arctan x + x| - \sin \arctan x + C
\]

To recover an elementary value for \( x \), sketch a right triangle with an angle \( \theta \) whose tangent is \( x \), find the missing third side of the triangle, and use this to find the other values:

From this we get \( \sin \arctan x = \frac{x}{\sqrt{x^2 + 1}} \), and \( \sec \arctan x = \sqrt{x^2 + 1} \), so the solution to the integral is

\[
\ln \left| \sqrt{x^2 + 1} + x \right| - \frac{x}{\sqrt{x^2 + 1}} + C.
\]

4. Integrate, using partial fraction decomposition:

\[
\int \frac{7}{(x - 1)(2x + 4)} \, dx
\]

We set up the sum as follows:

\[
\frac{7}{(x - 1)(2x + 4)} = \frac{A}{x - 1} + \frac{B}{2x + 4}, \quad \text{whence}
\]

\[
\frac{7}{(x - 1)(2x + 4)} = \frac{2Ax + 4A + Bx - B}{(x - 1)(2x + 4)}
\]

This gives us two equations to solve, \( 4A - B = 7 \), and \( 2A + B = 0 \). Solve these using elementary algebra to get \( B = -\frac{7}{3} \) and \( A = \frac{5}{6} \); this yields the integral

\[
\int \frac{7}{6} \left( \frac{1}{x - 1} - \frac{1}{x + 2} \right) = \frac{7}{6} \left( \ln |x - 1| - \ln |x + 2| \right) + C = \frac{7}{6} \ln \left| \frac{x - 1}{x + 2} \right| + C
\]