Show all work in the space provided or on scratch paper.

Determine whether each of the series converges (if it is alternating, check to see whether convergence is conditional or absolute). Give as much of an explanation as you can. Note the bonus question on the back page.

1. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n+1}}$

   This is a telescoping series with $n$th partial sum
   
   $$s_n = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}} + \cdots + \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n+1}}$$

   Thus, $\lim_{n \to \infty} s_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$

   and the series converges.

2. $\sum_{n=0}^{\infty} \frac{\cos(n\pi) \cdot 2^{2n+1}}{3^n}$

   This is a disguised alternating series, since $\cos(n\pi) = (-1)^n$. To check convergence, we consider the alternating series test, which essentially amounts to the same thing as performing the $n$th term test:

   $$\lim_{n \to \infty} \frac{2^{2n+1}}{3^n} = 2 \cdot \frac{4^n}{3^n} = \infty$$

   So this series fails the alternating series test and the $n$th term test, and thus diverges.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{\ln(n^{20} + 1)}$

   Again, for this case it makes the most sense to check the $n$th terms of the series to verify that they approach zero:

   $$\lim_{n \to \infty} \frac{n^2}{\ln(n^{20} + 1)} = \lim_{x \to \infty} \frac{x^2}{\ln(x^{20} + 1)}$$

   (by L'Hôpital’s rule) $\lim_{x \to \infty} \frac{2x}{20x^{19}}$

   $$= \lim_{x \to \infty} \frac{2x^{21} + 2x}{20x^{19}}$$

   $$= \infty$$

   Again, by the $n$th term test, the series is divergent.
4. $\sum_{n=1}^{\infty} \frac{2 \cdot n!}{(2n)!}$ Because of the factorial expressions, this series is probably best assessed through the use of the ratio test.

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2 \cdot (n + 1)! \cdot (2n)!}{(2n + 2)! \cdot 2 \cdot n!}$$

$$= \lim_{n \to \infty} \frac{n + 1}{(2n + 2)(2n + 1)}$$

$$= 0$$

With a ratio test limit of $\rho = 0$, we can say that the series is convergent.