Answer Key

Calculus II - Final Exam Review, Fall 2012 [Shaw]

1. Integrate:

(a) \[ \int \frac{4x - 1}{(x - 1)(x + 2)} \, dx = 3 \ln |(x + 2)(x - 1)| + C \] [partial fractions]

(b) \[ \int_{\pi/4}^{\pi} \sec^2(x) \tan(x) \, dx = -\frac{1}{2} \] [substitution with \( u = \tan(x) \)]

(c) \[ \int \frac{e^x}{1 - e^{2x}} \, dx = \frac{1}{2} \ln \left| \frac{1 + e^x}{1 - e^x} \right| + C \] [substitution with \( u = e^x \), followed by partial fractions]

(d) \[ \int_{0}^{1} (x - 1)^2 \ln(x) \, dx = \frac{1}{18} (11 - 9e^2 + 4e^3) \] [Integration by parts with \( u = \ln(x) \), and then some algebra on \( vdu \) making it easier to integrate]

(e) \[ \int x \cos(2x) \, dx = \frac{1}{2} x \sin(2x) + \frac{1}{4} \cos(2x) + C \] [Partial fractions with \( u = x \)]

(f) \[ \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx = 2 \sin(\sqrt{x}) + C \] [Substitution with \( u = \sqrt{x} \)]

(g) \[ \int_{3}^{\infty} e^{3-x} \, dx = 1 \] [Improper integral, substitution with \( u = 3 - x \)]

(h) \[ \int_{-\infty}^{-1} xe^{-x} \, dx \text{ DIVERGES} \] [Integrate first using parts, \( u = x \), and then see that \( \lim_{a \to -\infty} -xe^{-x} \) diverges to \( \infty \).]

(i) \[ \int_{1}^{\infty} \frac{1}{x^3 - x^2} \, dx \text{ DIVERGES} \] [Note that the function doesn’t exist at \( x = 1 \), so the integral is improper for both the top and bottom limit. You should end up, after partial fractions, with

\[ \ln \left| \frac{x - 1}{x} \right| + \frac{1}{x} \int_{1}^{\infty} \frac{1}{b} \ln \left| \frac{b - 1}{b} \right| + \frac{1}{b} = \lim_{a \to 1} \left( \ln \left| \frac{a - 1}{a} \right| - \frac{1}{a} \right) \]

All of these elements converge, except \( \lim_{a \to 1} \ln \left| \frac{a - 1}{a} \right| \), which diverges to \( -\infty \).]

2. Find the arc-length along the curve \( y = \frac{x^3}{21} + \frac{7}{4x} \) from \( x = 1 \) to \( x = 2 \).

The arc-length integral, \( \sqrt{1 + (f'(x))^2} \), works out to \( \frac{x^2}{7} + \frac{7}{4x^2} \). Once you integrate this from 1 to 2, you get \( \frac{29}{21} \).

3. Use any appropriate method to determine the volume of the solid generated by revolving the segment of the curve \( y = \sqrt{x + 1} \) from \( (0,1) \) to \( (7,2) \) about the y-axis.
You could use cylindrical shells or disks to answer this one; I use disks with radius \( r(y) = y^3 - 1 \) to get the integral

\[
\int_1^2 \pi (y^3 - 1)^2 \, dy = \frac{163\pi}{14}
\]

4. Find a solution to the differential equation, and to the initial value problem where applicable; if possible, write the solution as an explicit function of \( x \):

(a) \( \frac{dy}{dx} = x^3 y^4, \quad y(0) = 1 \)
\[
y = \left( -\frac{3}{4} x^4 + 1 \right)^{-1/3}
\]

(b) \( x e^y - \frac{dy}{dx} = 0 \)
\[
y = -\ln \left| -\frac{x^2}{2} + C \right|
\]

(c) \( \frac{dy}{dx} = y^2 - y \)
\[
y = \frac{1}{1+C e^x}
\]

(d) \( \frac{dy}{dx} = 12x^2 y - xy \)
\[
y = C e^{4x^3-x^2/2}
\]

5. The population of a small city in the rural midwest satisfies the exponential growth model. In this city, the population doubled from 1960 to 2010. What will the population be in 2030, relative to the 1960 population?
\[
A(70) = A_0 \cdot e^{70 \ln(2)}
\]

6. Determine whether each sequence with the given \( n^{th} \) term converges. If so, calculate the limit.

(a) \( a_n = \frac{n}{n^2 - 3n} \)

Converges to 0 [Check the leading terms]

(b) \( b_n = \frac{n(1 - e^n)}{e^{2n}} \)

Converges to 0 [Separate the factors, and use L’Hôpital on each]

(c) \( c_n = n^{20} \cdot 2^{-n} \)

Converges to 0 [L’Hôpital]

(d) \( d_n = \frac{\ln(\ln(n))}{\ln(n)} \)

Converges to 0 [L’Hôpital]
(e) $\alpha_n = \sin \left( \frac{1}{n} \right) \cdot e^n$

Diverges [Rewrite as a fraction $\frac{\sin \frac{1}{n}}{e^{-n}}$ and use L'Hôpital]

(f) $\beta_n = \frac{n^4 - \ln(2n)}{1 - n^4}$

Converges to $-1$ [Use L'Hôpital]

(g) $\gamma_n = \frac{\sqrt{n^4} - 1}{n^2}$

Converges to $1$ [Rewrite the limit as $\frac{\sqrt{\lim(f(x))}}{x}$ to simplify the inner limit; you then get $\sqrt{1} = 1$]

(h) $\delta_n = \frac{3^n}{n!}$

Converges to $0$ [Compare $3^n$ with $n!$ to see that $n!$ is eventually much larger.]

(i) $\varepsilon_n = \frac{\tan^{-1}(n)}{n}$

Converges to $0$ [Can't use L'Hôpital: in fact, the limit of $\tan^{-1}(n)$ is $\frac{\pi}{2}$]

7. Using any appropriate method, determine whether or not the series converges. You do not have to find the sum.

(a) $\sum_{n=1}^{\infty} \frac{2}{n}$

Convergent [Use the limit comparison test with $\frac{1}{n}$]

(b) $\sum_{n=1}^{\infty} \frac{10}{5^n}$

Convergent [Equivalent to the geometric series $\sum 10 \cdot \left( \frac{1}{25} \right)^n$]

(c) $\sum_{n=2}^{\infty} \left( \frac{2}{n-1} \right)^{1.001}$

Convergent [Limit comparison with $\left( \frac{1}{n} \right)^{1.001}$]

(d) $\sum_{n=1}^{\infty} \frac{2^n - 1}{7^n}$

Convergent [Rewrite as the difference of two geometric series: $\sum \left( \frac{2}{7} \right)^n - \sum \frac{1}{7^n}$]

(e) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \left( 1 - n \sqrt{\frac{1}{5}} \right)$

Convergent [Use the alternating series test and note that since $\sqrt{\frac{1}{5}}$ approaches $1$, the terms of the series approach $0$.]

(f) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$
Divergent [Use the comparison test; \( \frac{n}{\sqrt{n}} > \frac{1}{n} \); which diverges.]

\[ \sum_{n=1}^{\infty} \frac{\ln(5n)}{(n+2)^2} \]

Convergent [Use the limit comparison test with \( n^{-3/2} \), a convergent geometric series.]

\[ \sum_{n=1}^{\infty} \frac{n + 6^n}{n!} \]

Convergent [Use the ratio test.]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]

Convergent [Use the comparison test with \( \frac{1}{n^2} \)].

\[ \sum_{n=2}^{\infty} (-1)^n \cdot \frac{n - 1}{\ln(n)} \]

Divergent [Use the alternating series test or the nth term test and see that the terms of the series do not approach 0.]

\[ \sum_{n=1}^{\infty} \frac{2^n \cdot n!}{(2n)!} \]

Convergent [Ratio test.]

\[ \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^3 - 5}{n^4 + 2n - 1} \]

Convergent [Alternating series test]

\[ \sum_{n=1}^{\infty} (n!)e^{-n} \]

Divergent [Either the nth term test or the ratio test, which will give you a ratio approaching \( \infty \).]

8. Find the sum:

(a) \[ 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots = \sum_{n=0}^{\infty} \left( -\frac{2}{3} \right)^n = \frac{1}{1 + \frac{2}{3}} = \frac{3}{5} \]

(b) \[ \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=0}^{\infty} \frac{3}{10^n} \cdot \left( \frac{1}{10} \right)^n = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{30} \]

9. For each of the series below, find the radius and interval of convergence for \( x \).

(a) \[ \sum_{n=0}^{\infty} (x - 3)^n \]

\[ R = 1; \text{ the interval of convergence is } (2, 4). \]
(b) \( \sum_{n=0}^{\infty} \frac{2^n}{n!} (x)^n \)

\[ R = \infty; \text{ the series converges on all of } \mathbb{R}. \]

(c) \( \sum_{n=0}^{\infty} \left( \frac{x^2 + 5}{5} \right)^n \)

This isn’t a power series, in fact: when you perform the ratio test you get that \( x \) must satisfy \( \left| \frac{x^2 + 5}{5} \right| < 1 \), which is impossible; thus the radius of convergence is 0 and the interval of convergence is \( \emptyset \).

(d) \( \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n+1} \)

\[ R = 1; \text{ the interval of convergence is } (-1, 1). \]

(e) \( \sum_{n=0}^{\infty} (e^x - 4)^n \)

\[ R = \ln(\sqrt{5}); \text{ the interval of convergence is } (\ln(3), \ln(5)). \]