**Calculus I - A Note on Cones in Spheres**

This note addresses problem 12 from the book in section 4.6, the chapter on applied optimization. The problem asked for the dimensions of the cone with largest volume that can be inscribed in a sphere of radius 3. When solving this problem on the blackboard for the first time, I made a devastating tactical error, which had dire consequences, but which is quite simple to repair. Using the nomenclature from the textbook ($x$ and $y$ rather than $r$ and $y$), recall that the volume formula for this cone will be

$$V = \frac{1}{3}\pi x^2(y+3);$$

and also that the relationship between $x$ and $y$ given by the Pythagorean theorem is

$$x^2 + y^2 = 9$$

On that occasion, when doing this problem on the board, I chose to solve for $y$ and then proceed by evaluating the equation:

$$V = \frac{1}{3}\pi x^2\left(\sqrt{9-x^2}+3\right)$$

This was probably due to my own ingrained tendency to like working with $x$ rather than $y$, and while mathematically correct, this led to a difficult derivative with several pratfalls involved in solving for the critical points. The much more sensible choice in this case would have been to solve for $x$, and get $x = \sqrt{9-y^2}$:

$$V = \frac{1}{3}\pi\left(\sqrt{9-y^2}\right)^2(y+3) = \frac{1}{3}\pi\left(-y^3-3y^2+9y+27\right)$$

Note that this is just a polynomial, and as such the derivative will be quite a bit easier to find and set to zero. The lesson here is that you should make sure to look at your substitution before continuing the problem, to be sure you made a good decision about which variable to work with.

Finishing the problem, we find:

$$V'(y) = \frac{1}{3}\pi\left(-3y^2-6y+9\right) = -\pi(y^2+2y-3)$$

and setting this equal to zero subject to the domain gives $y = 1$. If you plug this into the equation $x = \sqrt{9-y^2}$, you get $x = \sqrt{8}$, which means the radius of the cone will be $\sqrt{8}$ and the height will be $1+3=4$.

For your edification and entertainment, I will repeat here the steps that we would take to find the solution by substituting for $y$ instead. Recall that the equation we got was

$$V = \frac{1}{3}\pi x^2\left(\sqrt{9-x^2}+3\right) = \frac{1}{3}\pi\left(x^2\sqrt{9-x^2}+3x^2\right)$$

We use this form to find $V'(x)$, using the product rule and chain rule where appropriate:

$$V'(x) = \frac{1}{3}\pi\left(2x\sqrt{9-x^2}+\frac{1}{2}x^2\cdot(-2x)\cdot(9-x^2)^{-1/2}+6x\right)$$

$$= \frac{1}{3}\pi\left(2x\sqrt{9-x^2}-\frac{x^3}{\sqrt{9-x^2}}+6x\right) = \frac{1}{3}\pi x\left(2\sqrt{9-x^2}-\frac{x^2}{\sqrt{9-x^2}}+6\right)$$

Since we are solving when this expression is equal to zero, and we are not considering the case where $x = 0$, we may delete off the outer term to get the equation:

$$2\sqrt{9-x^2}-\frac{x^2}{\sqrt{9-x^2}}+6 = 0$$

Now the domain of the function is $[0,3]$, so for calculating the derivative, we can assume $x \neq \pm 3$, we may multiply through by $\sqrt{9-x^2}$ to clear the denominators:

$$0 = 2\left(9-x^2\right)-x^2+6\sqrt{9-x^2} = 18-3x^2+6\sqrt{9-x^2}$$

A little bit of a trick will be to treat this like a quadratic equation in the variable $\sqrt{9-x^2}$, which means we have to make all of our nonconstant terms look like powers of $\sqrt{9-x^2}$. Subtract and add 9, and rewrite as:

$$0 = -9 + 9 + 18 - 3x^2 + 6\sqrt{9-x^2} = -9 + (27 - 3x^2) + 6\sqrt{9-x^2} = -9 + 3\left(9 - x^2\right) + 6\sqrt{9-x^2}$$

Now, the quadratic substitution. Let $u = \sqrt{9-x^2}$. Then the equation becomes:

$$0 = -9 + 3u^2 + 6u, \text{ or } 3(u^2 + 2u - 3) = 3(u+3)(u-1) = 0$$

The negative solution is inadmissible, and the positive solution is $u = 1$, which corresponds to $\sqrt{9-x^2} = 1$, giving $x = \sqrt{8}$. A quick check reveals that in this case $y = 1$, exactly as in the first method.

I trust you will find this problem much simpler to do in the first way.

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