10.4: Applications of linear differential equations

1. Audrina plans to ruin LC’s dream of starting her own clothing line, but first she needs to make enough money to buy a flamethrower. Audrina opens a bank account account with $1,000, earning 3.5% annual interest compounded continuously, and from then on makes daily deposits totaling $7,000 per year. Write a formula $P(t)$ which represents the amount of money in the account, $t$ years after Audrina opens it.

This is similar to the simple interest problem we studied in Math 220. The bank account will be changed by the growing interest (increase) and the continuous deposits (also increase). So, the differential equation will be

$$y' = \text{interest} + \text{annual continuous deposit}$$

$$y' = .035y + 7000$$

And the initial deposit of $1,000 means that we have the initial condition $y(0) = 1000$. Solve this linear IVP to get equation $P(t)$.

2. Comic book superhero The Question got fired from her job as a police detective, $t+y+1=0$

Solve this linear IVP to get equation $f(t)$.

2. Comic book superhero The Question got fired from her job as a police detective, and needs to continue to buy gadgets, so she borrows $40,000 from the Penguin, who charges her 25% interest. If she makes continuous payments totalling $15,000 per year, how long until she pays off the loan?

Let $P(t)$ represent the balance of the loan at time $t$. In this case, the balance goes up with the interest, but goes down with the payments! So the differential equation is:

$$y' = \text{interest} - \text{annual continuous payment}$$

$$y' = .25y - 15000,$$ with initial condition $y(0) = 40000$.

We solve this linear IVP to get $P(t)$, and set $P(t) = 0$ to find when the balance = 0.

3. According to the novel “The Heat of the Heart,” whenever Fabio is near to her, Marianita’s body temperature rises 5 degrees Celsius. At work, where the ambient temperature is 30°C, Marianita had a normal body temperature of before 37°C before running into Fabio. Suppose the constant of proportionality for Marianita’s love-stricken body is $k = .2$. How long after leaving Fabio until Marianita has her normal body temperature again?

Newton’s law of cooling says that the change in temperature is proportional to the difference between the surrounding area and the temperature of the object (in this case, Marianita herself). So the differential equation is:

$$y' = k(30 - y),$$ where $k = .2$

Since Marianita has just left Fabio, her starting temperature (initial condition) is $y(0) = 37 + 5 = 42$.

Now we have a linear IVP, which we can solve to get a formula $f(t)$ for Marianita’s temperature at time $t$, and then we set $f(t) = 37$ and solve for $t$ to answer the question.

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**Spring break mathematics reading**

While on the beach, place this booklet in the center of an US Weekly magazine, a comic book, or a trashy romance novel, and proceed as if you are really enjoying a nice relaxing vacation away from mathematics. Your friends will never suspect that you are really studying. This booklet is a review of the major components of Chapter 10 so far.

- Section 10.1: Solutions of differential equations
- Section 10.2: Separation of variables
- Section 10.3: First-order linear differential equations
- Section 10.4: Applications of linear differential equations

**Chapter 10.1 - Solutions of Differential Equations**

1. What are all of the constant solutions for $y' = 3ty^2 - 6ty$?

A constant solution of a differential equation is a solution $y = c$ for some constant $c$. To find whether or not there is a constant solution, set $y' = 0$ and solve for $y$. This often involves factoring.

$$y' = 0 = 3ty(y - 2),$$ so $y = 2$ or $y = 0$.

2. Draw a direction field for the equation $y' = t + y + 1$, $-2 \leq t \leq 2$, $-2 \leq y \leq 2$.

A direction field is a graph on the $ty$-plane of all of the possible values for $y'$. Remember that $y'$ is the slope of the solution curve $y$. We draw the direction field using a chart of values, with $y'$ as the chart entries.

3. Check to see whether $f(t) = \sin(2t) + 5\cos(2t)$ is a solution to the initial value problem $y'' + 4y = 0$, $y\left(\frac{\pi}{2}\right) = -5$.

To check this, we have to check both of the equations. Plug $f(t)$ in for $y$, $f'(t)$ in for $y'$, $f''(t)$ in for $y''$, etc. And for the initial condition $y\left(\frac{\pi}{2}\right) = -5$, we plug $\frac{\pi}{2}$ for $t$ and $-5$ in for $y$. First, we need to find the second derivative, $f''(t) = -4\sin(2t) - 20\cos(2t)$. (Next page)
Chapter 10.2 - Separation of Variables

Separation of variables allows us to solve differential equations which can be written in the special form \( y' = p(t)g(y) \), where \( p \) is a function whose only variable is \( t \), and \( g \) is a function whose only variable is \( y \). Sometimes we have to do some manipulation to get a differential equation into standard form. Then we divide by \( q(y) \) and integrate both sides with respect to \( t \), and solve for \( y \), to get the general solution.

1. Solve \( y' = \frac{t^2+2}{5\sin(3t)} \).

2. Solve the IVP \( y' = e^{t+y}, \ y(0) = 0 \).

In order to use separation of variables, we have to see that we can rewrite the right side of the equation to get \( y' = e^{t} \cdot e^{y} \). Now we solve as before:

\[
\begin{align*}
\int e^{-y} \frac{dy}{dt} & = e^t \\
\int e^{-y} dy & = \int e^t dt \\
e^{-y} & = e^t + C_0 \\
e^{-y} & = -e^t - C_0; \ C = -C_0 \\
\ln(e^{-y}) & = 1 = 1 + C \\
y & = -\ln(-e^t + C)
\end{align*}
\]

Chapter 10.3 - First-Order Linear Differential Equations

A First Order Linear Differential Equation (FOLDE) is an equation which can be written in the “standard form” \( y' + a(t) \cdot y = b(t) \).

To solve a FOLDE, we first have to put it into standard form, and identify \( a(t) \) and \( b(t) \). Next we find \( A(t) = \int a(t) dt \). We form the integrating factor \( e^{A(t)} \), and multiply both sides of the equation by the integrating factor, to get

\[ e^{A(t)} y' + e^{A(t)} \cdot a(t) \cdot y = e^{A(t)} \cdot b(t) \]

Once the equation is in this form, we can recognize (by remembering the product rule) that the left side is the derivative of the product \( e^{A(t)} \cdot y \). (Think \( f = y, g = e^{A(t)} \), and the product rule says the derivative of \( fg \) is \( f'g + fg' \), which is the left side of the equation.) So we rewrite the equation as:

\[ \frac{d}{dt} \left[ e^{A(t)} y \right] = e^{A(t)} \cdot b(t) \]

Integrating both sides, we get:

\[
\int \frac{d}{dt} \left[ e^{A(t)} y \right] dt = \int e^{A(t)} \cdot b(t) dt
\]

1. Solve the IVP \( ty' - y = -1, \ y(1) = 1, \ t > 0 \)

First we put it in standard form by dividing by \( t \).

\[
y' - \frac{y}{t} = -\frac{1}{t}
\]

So \( a(t) = -\frac{1}{t} \) and \( b(t) = -\frac{1}{t} \), and thus \( A(t) = -\ln(t) \). (We can forget about the absolute value since \( t > 0 \).) Thus the integrating factor \( e^{A(t)} = e^{\ln(t)} = 1 \). So our new equation is back to the original! Now if we look closer we see that in fact the left side is the derivative of \(-ty\).

\[
\frac{d}{dt} [-ty] = -1
\]

Integrate, solve for \( y \), and solve for \( C \):

\[
\int \frac{d}{dt}[-ty] = \int -1 dt
\]

\[
-ty = -\frac{1}{2} t + C_0; \ C = -C_0 \\
y = \frac{1}{2} + \frac{C}{t}
\]