April 24, 2009 Solutions to Practice - Math 221 section 02xx Exam 3 [Shaw]

1. (a) Remember that Newton-Raphson is only used to estimate which values of x make a certain function equal to zero. So, to find when $x = \sqrt{10}$, we need to use the function $f(x) = x^2 - 10$, which is zero when $x = 10$. Note also that $f'(x) = 2x$. We are told $x_0 = 3$, so we calculate:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3^2 - 10}{(2)(3)} = 3 - \frac{-1}{6} = \frac{19}{6}$$

$$x_2 = \frac{19}{6} - \frac{\left(\frac{19}{6}\right)^2 - 10}{2 \cdot \left(\frac{19}{6}\right)}$$

And we stop there, since $n = 2$, and the problem says not to simplify.

(b) i. To use the formula for calculating a Taylor polynomial about $x = a$, we need to know the derivatives $\sin(x)$ for the value $x = \left(\frac{\pi}{2}\right)$.

$$\begin{array}{lll} f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) \\ f\left(\frac{\pi}{2}\right) = 1 & f'\left(\frac{\pi}{2}\right) = 0 & f''\left(\frac{\pi}{2}\right) = -1 \end{array}$$

So, the Taylor polynomial is:

$$p_2(x) = 1 + \frac{0}{1!} \cdot \left(x - \frac{\pi}{2}\right) + \frac{-1}{2!} \cdot \left(x - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

ii. Thus, $p_2\left(\frac{11\pi}{20}\right) = 1 - \frac{1}{2}\left(\frac{11\pi}{20} - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2}\left(\frac{\pi}{20}\right)^2 = \frac{800 - \pi^2}{800}$

2. (a) $1.\overline{23} = 1 + \left(\frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \cdots\right)$

The part in the parentheses is a geometric series with $a = \frac{23}{100}$, and $r = \frac{1}{100}$. Thus the whole answer is $1 + \frac{\frac{23}{100}}{1 - \frac{1}{100}} = 1 + \frac{\frac{23}{100}}{\frac{99}{100}} = 1 + \frac{23}{99}$, or $\frac{122}{99}$.

(b) $\sum_{n=0}^{\infty} 2\left(\frac{2}{7}\right)^n$ is a geometric series, which in this case is easy to interpret: if we write out the first few terms, we get $2 + 2 \cdot \frac{2}{7} + 2 \cdot \left(\frac{2}{7}\right)^2 + \cdots$. We can find that $a = 2$ and $r = \frac{2}{7}$, so the sum is $\frac{2}{1 - \frac{2}{7}} = \frac{2}{\frac{5}{7}} = \frac{14}{5}$.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{1}{n}\right)$ is not a geometric series, so we need to use the integral test, and find out whether or not $\int_{n=1}^{\infty} \frac{1}{x^2} \sin\left(\frac{1}{x}\right)$ is convergent. This integral is a u-substitution with $u = \frac{1}{x}$. It turns out to be convergent, with value equal to:

$$\lim_{t \to \infty} \cos\left(\frac{1}{t}\right) - \cos(1) = \cos(0) - \cos(1)$$

Even though we don't know what $\cos(1)$ is, we know this means the series is convergent.

4. (a) This one involves the Taylor series for $\dfrac{1}{1-x}$. The function $\dfrac{x^2}{1+x^3}$ can be rewritten as $x^2 \cdot \dfrac{1}{1-(-x^3)}$, so we have a substitution and a multiplication problem:

$$x^2 \cdot \frac{1}{1-(-x^3)} = x^2\left(1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + (-x^3)^4 + \cdots\right)$$

$$= x^2 - x^5 + x^8 - x^{11} + x^{14} - \cdots$$

(b) If $g(x) = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^4 - \dfrac{1}{4!}x^6 + \dfrac{1}{5!}x^8 - \cdots$, then $\displaystyle\int 2g(x)\,dx = 2\left(x - \frac{1}{3\cdot 2!}x^3 + \frac{1}{5\cdot 3!}x^5 - \frac{1}{7\cdot 4!}x^7 + \frac{1}{9\cdot 5!}x^9 - \cdots\right) + C$. Setting $G(0) = 2$ gives us $C = 2$, so
$G(x) = 2 + 2x - \dfrac{2}{3\cdot 2!}x^3 + \dfrac{2}{5\cdot 3!}x^5 - \dfrac{2}{7\cdot 4!}x^7 + \dfrac{2}{9\cdot 5!}x^9 - \cdots$

5. (a) $E(x) = 0\cdot\dfrac{12}{30} + 1\cdot\dfrac{11}{30} + 2\cdot\dfrac{5}{30} + 3\cdot\dfrac{2}{30} = \dfrac{27}{30}$

(b) $V(x) = \dfrac{12}{30}\left(0 - \dfrac{27}{30}\right)^2 + \dfrac{11}{30}\left(1 - \dfrac{27}{30}\right)^2 + \dfrac{5}{30}\left(2 - \dfrac{27}{30}\right)^2 + \dfrac{2}{30}\left(3 - \dfrac{27}{30}\right)^2$

(c) Finally, the standard deviation $\sigma(x) = \sqrt{V(x)}$.