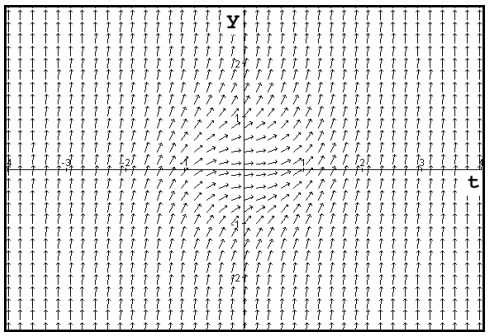
April 1, 2009 **Math 221 section 02xx Exam 2 [Shaw]**

No calculators allowed. You must show work at all times in order to receive full credit.
Do questions 1-2 on Sheet 1, 3-4 on Sheet 2, and 5-6-7 on Sheet 3.

1. **[5pts]** Which of the following is the differential equation whose direction field is represented in the figure to the right? *Give a one-sentence explanation why.*



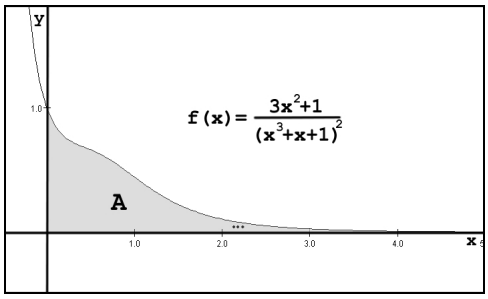
   a. $y' = 2y - 1$
   b. $y' = t^2 - 2$
   c. $y' = y^2 + t^2$
   d. $y' = -ty$

2. For this question, use the ecological growth equation $\dfrac{dN}{dt} = \dfrac{r}{K}N(K - N)$.
   The huge fish tank at Hank's pet store has a large population of guppies. The carrying capacity of the tank is 1000 guppies, and when there are 700, they are growing at a rate of 50 *per month*.
   (a) **[5pts]** Write a logistic differential equation satisfied by $N(t) =$ the number of guppies *after* t *months*. Hint: first solve for r using the information in the problem.
   (b) **[10pts]** Draw the $yz-$graph corresponding to your equation, and then on the $ty-$graph, sketch the constant solution(s) and solutions corresponding to $N(0) = 800$ and $N(0) = 1200$.
   (c) **[5pts]** What population level corresponds to the maximum growth rate?

3. For each of the following initial value problems, find both the general solution and the particular solution, using any appropriate technique from class:
   (a) **[15pts]** $y' = 3y^2e^t + y^2;\ y(0) = -1$
   (b) **[15pts]** $ty' - t^2 = -ty;\ y(0) = 0$

4. Beatrice is a celebutante with a large trust fund from her parents. The trust fund earns 5% annual interest compounded continuously. Beatrice doesn't have a job so her 'income' consists of continuous withdrawals from her trust fund, which come out to \$2 million per year. Let D represent the initial deposit in the trust fund.
   (a) **[5pts]** Write an initial value problem satisfied by $P(t) =$ the amount, in millions, in her trust fund after t years. (Hint: $P(0)$ is the initial deposit in the account.)
   (b) **[5pts]** What is the smallest value of D so that the account *does not lose* money over time?

5. **[15pts]** Construct and solve an improper integral expressing the area of the shaded region A in the figure to the right. Keep in mind that A should be assumed to continue infinitely to the right. If the area is infinite (*i.e.* the integral diverges), write *DNE*.



6. **[15pts]** Show that $f(t) = e^{4t} - 3$ is a solution to the differential equation $y'' - 3y' - 4y - 12 = 0$.

7. **[10pts]** Let $f(t)$ be the solution to the IVP $y' = \dfrac{y^2}{t+1};\ y(0) = 1$. Use Euler's method with $n = 2$ on $0 \le t \le 1$ to estimate $f(1)$.

Math 221 section 02xx Exam 2 [Shaw] **Solutions**

Note: Please remember to add dt or dx or dy or whatever the appropriate variable is when taking integrals. I did not take off points on the problems I graded, but you *must* do this on all remaining exams and quizzes to get full credit.

1. a. $\cancel{y' = 2y - 1}$

   b. $\cancel{y' = t^2 - 2}$

   c. $y' = y^2 + t^2$ ✓

   d. $\cancel{y' = -ty}$

   Reason: The slope of the solutions are all greater than or equal to zero, and the only choice with that property is choice *c*.

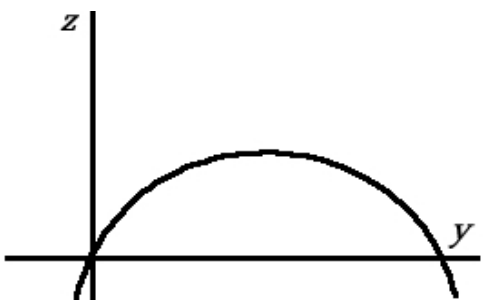
2. (a) In the logistic ecological growth model, the value of K is the carrying capacity, so we just need to solve for r. The problem says that when N is 700, then $\frac{dN}{dt}$ is 50. So:

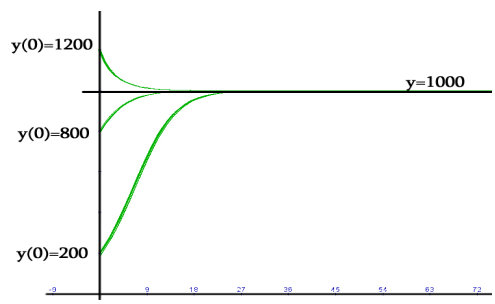
$$50 = \frac{r}{1000} \cdot 700 \cdot (1000 - 700) = \frac{700r}{1000} \cdot 300 = 210r$$

$$r = \frac{50}{210} = \frac{5}{21}$$

$$\text{so the equation is } \frac{dN}{dt} = \frac{5}{21000} N(1000 - N)$$

(b)





(c) The maximum growth rate corresponds to $y = 500$; if you look at the solution $y(0) = 200$, you can see the inflection point occurring at about $y = 500$.

3. (a) $y' = 3y^2e^t + y^2$; $y(0) = -1$. The first question is not linear since there is a y^2, so it must be separable. We first have to factor the right side, and then it becomes pretty straightforward. Remember that for an initial value problem, we need to find both the general solution and the particular solution.

$$\frac{dy}{dt} = y^2(3e^t + 1)$$

$$\int y^{-2}\, dy = \int 3e^t + 1\, dt$$

$$-y^{-1} = 3e^t + t + C$$

$$\frac{1}{y} = -3e^t - t + C$$

$$y = \frac{1}{-3e^t - t + C}$$

$$y(0) = -1$$

$$-1 = \frac{1}{-3 + C}$$

$$-3 + C = -1$$

$$C = 2$$

Particular solution: $y = \dfrac{1}{-3e^t - t + 2}$

(b) $ty' - t^2 = -ty$; $y(0) = 0$. This is a linear equation, so we first have to put it into standard form, identify a and b, and then multiply through by the integrating factor.

$$ty' + ty = t^2$$
$$y' + y = t$$
$$a(t) = 1$$
$$A(t) = t$$
$$b(t) = t$$

$$y'e^t + ye^t = te^t$$
$$\frac{d}{dt}\left[ye^t\right] = te^t$$
$$\int \frac{d}{dt}\left[ye^t\right]\,dt = \int te^t\,dt$$
$$ye^t = te^t - e^t + C \text{ (int. by parts)}$$
$$y = t - 1 + Ce^{-t}$$

$$y(0) = 0$$
$$0 = 0 - 1 + C$$
$$C = 1$$

Particular solution: $y = t - 1 + e^{-t}$

4. (a) $y' = .05y - 2$; $y(0) = D$. **Note:** You need *both* parts for an IVP, but since less than 30 students remembered to put the initial conditions down as part of the problem, instead of taking off points for the rest, I added a point to those who *did* put in the initial conditions.

(b) The smallest value of D for which you don't lose money is the constant solution!

$$0 = .05y - 2$$
$$.05y = 2$$
$$y = \frac{2}{.05} = \frac{2}{\frac{1}{20}} = 2 \cdot 20$$
$$= \$40 \text{ million}$$

Note: By a technicality, I also gave full credit to anyone who wrote down $D = 0$... since if the bank account doesn't have any money, then the celebutante can't take any money out of it, so it can't lose money. But this was *not* the intended answer for the question.

5. You should be able to write down the integral based on the picture:

$$\int_0^\infty \frac{3x^2 + 1}{(x^3 + x + 1)^2}\,dx$$

Once you have the integral, it's an easy substitution, which I suggested in previous cases that you do in a separate area so you don't get confused:

$$\int \frac{3x^2 + 1}{(x^3 + x + 1)^2}\,dx = \int u^{-2}\,du = -u^{-1}$$

$u = x^3 + x + 1;\ du = (3x^2 + 1)dx$

Then we plug in for the values:

$$\lim_{t\to\infty} \frac{-1}{x^3 + x + 1}\Bigg|_0^t = \lim_{t\to\infty} \frac{-1}{t^3 + t + 1} - \frac{-1}{0 + 0 + 1} = 0 + 1 = 1$$

6. Show that $f(t) = e^{4t} - 3$ is a solution to the differential equation $y'' - 3y' - 4y - 12 = 0$.

$y = e^{4t} - 3$
$y' = 4e^{4t}$
$y'' = 16e^{4t}$

$y'' - 3y' - 4y - 12 \stackrel{?}{=} 0$
$16e^{4t} - 3(4e^{4t}) - 4(e^{4t} - 3) - 12 \stackrel{?}{=} 0$
$16e^{4t} - 12e^{4t} - 4e^{4t} + 12 - 12 = 0$✓

7. $g(t, y) = \frac{y^2}{t+1}$; $t_0 = 0$; $y_0 = 1$; $n = 2$, so $h = \frac{1}{2}$.

$t_1 = \frac{1}{2} \quad y_1 = 1 + \frac{1}{2}\left(\frac{1}{0+1}\right) = 1 + \frac{1}{2} = \frac{3}{2}$

$$t_2 = 1 \quad y_2 = \frac{3}{2} + \frac{1}{2}\left(\frac{\left(\frac{3}{2}\right)^2}{\frac{1}{2} + 1}\right) = \frac{3}{2} + \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{3}{2}}\right) = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$