Exam 2 Practice – Solutions

No calculators allowed. You must show work at all times in order to receive full credit.

1. *Use logarithmic differentiation to find the derivative:*

$$f(x) = \frac{e^{3x}(x+1)^6(x^3-2)^4}{\sqrt{2x-7}}$$

Recall that logarithmic differentiation relies on the following:

$$\frac{d}{dx}\left[ln(f(x))\right] = \frac{f'(x)}{f(x)}, \text{ so}$$

$$f'(x) = f(x) \cdot \frac{d}{dx}\left[ln(f(x))\right], \text{ which is easier to find.}$$

So first we find $\frac{d}{dx}\left[ln(f(x))\right]$:
$\ln\left(\frac{e^{3x}(x+1)^6(x^3-2)^4}{\sqrt{2x-7}}\right) =$
$\ln(e^{3x}) + 6\ln(x+1) + 4\ln(x^3-2) - \frac{1}{2}\ln(2x-7)$. Thus:

$$f'(x) = f(x) \cdot \frac{d}{dx}\left[ln(f(x))\right] = \left(\frac{e^{3x}(x+1)^6(x^3-2)^4}{\sqrt{2x-7}}\right)\left(3 + \frac{6}{x+1} + \frac{4 \cdot 3x^2}{x^3-2} - \frac{1}{2} \cdot \frac{2}{2x-7}\right)$$

2. (a) *Find* all *values for x which make the equation true:* $x\ln(x+1) = 2x$
To solve, we need to set the summands equal to zero and factor:

$$x\ln(x+1) - 2x = 0$$

$$x(\ln(x+1) - 2) = 0$$

So, the solutions are $\boxed{x = 0}$ and $\ln(x+1) - 2 = 0$, which we solve as follows:

$$\ln(x+1) = 2$$

$$e^{\ln(x+1)} = e^2$$

$$x + 1 = e^2, \text{ so:}$$

$$\boxed{x = e^2 - 1}$$

(b) *Write the following expression in the form 2^{ax+b} for some a and b:* $\frac{16^x}{2 \cdot 4^{2x}}$

$$\frac{16^x}{2 \cdot 4^{2x}} = \frac{(2^4)^x}{2 \cdot (2^2)^{2x}} = \frac{2^{4x}}{2 \cdot 2^{4x}} = \frac{1}{2} = 2^{-1}.$$

3. *Find $g'(1)$, where* $g(x) = e^{3x^2-2x+1}$

$$g'(x) = (6x-2) \cdot e^{3x^2-2x+1}, \text{ so } g'(1) = (6-2) \cdot e^{3-2+1} = 4e^2$$

4. *For the following function, determine the x- and y-value of each critical point, and then use the first or second derivative test to determine whether each point is a maximum, minimum, or neither:*

$$f(x) = \frac{x}{\ln(x)}$$

$$f'(x) = \frac{1 \cdot \ln(x) - x \cdot \frac{1}{x}}{(\ln(x))^2} = \frac{\ln(x) - 1}{(\ln(x))^2}$$

$$\text{Solving for zero, we get } \ln(x) - 1 = 0$$

$$\ln(x) = 1$$

$$x = e$$

The first derivative test will have the interval on the left $x < e$ and the interval on the right $x > e$. Since the function is not defined for zero or for 1, we will check the value of the derivative for $\sqrt{e} = e^{\frac{1}{2}}$ and e^2:

$$f'(e^{\frac{1}{2}}) = \frac{\frac{1}{2} - 1}{\left(\frac{1}{2}\right)^2} < 0$$

$$f'(e^2) = \frac{2 - 1}{(2)^2} > 0$$

So by the first derivative test, there is a maximum at $x = e$, and the y-value at this point is $f(e) = \frac{e}{1} = e$. So the maximum occurs at the point (e, e).

5. *To make killer robots, one uses the radioactive isotope* `Halloweenium-X`*, which has a decay constant of* $\lambda = .002$.

   (a) *What is the half-life of* `Halloweenium-X`*?*
   Recall that $L = \frac{\ln(2)}{\lambda}$, so in this case the half-life is $L = \frac{\ln(2)}{.002}$.

   (b) *What differential equation is satisfied by the decay of* `Halloweenium-X`*?*
   The differential equation for exponential decay is $P'(t) = -\lambda P(t)$, so $P'(t) = -.002P(t)$.

   (c) *Use the differential equation to answer the question: how much* `Halloweenium-X` *is in a sample which is decaying at the rate of 4 grams per year?*
   This problem is asking the question: what is $P(t)$ when $P'(t) = -4$? We answer by plugging the quantity into the equation from part b: $-4 = -.002 \cdot P(t)$, so $P(t) = \frac{-4}{-.002} = 2000$ grams.

6. *A colony of flesh-eating zombies feeds on the living population of Earth and grows at a rate proportional to its size. (As with all zombie colonies) it starts with just one person, and after one year there are 2,500 zombies.*

   (a) *Find the growth constant k of the zombie colony.*
   The equation is $P(t) = P_0 e^{kt}$, and the problem tells us $P_0 = 1$ and $P(1) = 2500$. So we solve for k as follows:

$$2500 = 1 \cdot e^{k \cdot 1}$$

$$\ln(2500) = k$$

   (b) *Find an equation $P(t)$ which describes the number of zombies in the colony at time t.*
   This is straightforward enough. $P(t) = e^{(\ln(2500)) \cdot t}$

(c) *Using this equation, how many years until the colony reaches 10,000 zombies?*
Solve the equation for t:

$$P(t) = 10000 = 1 \cdot e^{(\ln(2500)) \cdot t}$$

$$\ln(10000) = (\ln(2500)) \cdot t$$

$$t = \frac{\ln(10000)}{\ln(2500)}$$

7. *An evil collector sells his collection of cursed talismans for* $\$4,000$, *deposits the earnings into his bank account, then abruptly disappears and is never heard from again. The account accrues interest compounded continuously for 100 years before someone discovers it, with a balance of* $\$120,000$. *What was the annual interest rate for this bank account?*
We know that the principal P is 4000, and we're given that $A(100) = 120000$. So, we solve for r:

$$A(t) = Pe^{rt}$$

$$A(100) = 120000 = 4000e^{r \cdot 100}$$

$$\frac{120000}{4000} = 30 = e^{r \cdot 100}$$

$$\ln(30) = 100r$$

$$r = \frac{\ln(30)}{100}$$

8. *The concentration of a drug in the bloodstream of a patient, t hours after injection, is given by:*

$$f(t) = 5(e^{-.2t} - e^{-2t}) \textit{ units}$$

At what rate is the drug concentration changing after 4 hours?
We just need to find $f'(4)$, and this is a derivative we should be able to manage.

$$f'(t) = 5 \cdot (-.2e^{-.2t} + 2e^{-2t})$$

$$f'(4) = 5 \cdot (-.2e^{-.8} + 2e^{-8})$$