Weakly o-minimal structures and Skolem functions

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Logic Seminar
The George Washington University

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1. Background
   - Brief History
   - Model - definable - QE - DNF - o-minimal - Skolem function
   - Weakly o-minimal structures
   - Monotonicity

2. Valuational and nonvaluational cuts
   - Motivation
   - Definable subgroups
   - Good news
   - Pathologies

3. Skolem functions in valuational structures
   - $T$-immunity
   - Technique for elimination
   - Corollaries
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Catching up on the timeline

- 332 BC
- 1879
- 1900
- 1910
- 1920
- 1930
- 1931
- 1937
- 1949
- 1984
- 1996
- 2000
- 2008

Weakly o-minimal structures and Skolem functions
322 BC - *Organon*, Aristotle
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- “The categories”
- “On interpretation”
- “The prior analytics”
- “The posterior analytics”
- “The topics”
- “Sophistical refutations”
“Modern logic”

1879 - *Begriffsschrift*, Gottlob Frege
“Naive set theory”
1900 - Hilbert’s problems
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1901 - Russell’s Paradox
1910 - *Principia Mathematica*, Bertrand Russell and Alfred Whitehead
Type theory
382 PROLEGOMENA TO CARDINAL ARITHMETIC [PART 11

\*54:56. \(\vdash \alpha \sim \epsilon 0 \cup 1 \cup 2. \equiv (\exists x, y, z). x, y, z \in \alpha. x \neq y. x \neq z. y \neq z\)

Dem.

\(\vdash \alpha \sim \epsilon 0 \cup 1 \cup 2. \equiv (\exists x, y). x, y \in \alpha. x \neq y. \alpha \equiv t'x \cup t'y:\)

\text{[*51:2.*22:59]} \equiv (\exists x, y). t'x \cup t'y \subset \alpha. x \neq y. \alpha \equiv t'x \cup t'y:\)

\text{[*24:6]} \equiv (\exists x, y). t'x \cup t'y \subset \alpha. x \neq y. \exists ! \alpha - (t'x \cup t'y):\)

\text{[*51:232.Transp]} \equiv (\exists x, y). t'x \cup t'y \subset \alpha. x \neq y. \exists ! (\exists z). z \in \alpha. z \neq x. z \neq y:

\text{[*51:2.*22:59]} \equiv (\exists x, y, z). x, y, z \in \alpha. x \neq y. x \neq z. y \neq z. \vdash \alpha \vdash \text{Prop.}\)

In virtue of this proposition, a class which is neither null nor a unit class nor a couple contains at least three distinct members. Hence it will follow that any cardinal number other than 0 or 1 or 2 is equal to or greater than 3. The above proposition is used in \*104:43, which is an existence-theorem of considerable importance in cardinal arithmetic.

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Weakly o-minimal structures and Skolem functions
1920 - Thoralf Skolem refines the proof of Löwenheim’s theorem on model-existence.
Beginnings of computability theory

1930 - Completeness theorem
1931 - Incompleteness theorems
1930 - Tarski-Seidenberg theorem shows the real field, $\mathcal{R} = (\mathbb{R}, +, \cdot, <, 0, 1)$, eliminates quantifiers (and thus in later parlance is decidable).
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Beginnings of computability theory

1937 - Turing machine
1949 - Julia Robinson shows that in the rational field \( \mathbb{Q} = (\mathbb{Q}, +, \cdot, <, 0, 1) \) one can define the positive integers.
1980s - o-minimal structures - definition and key classification results
1996 - Wilkie showed that $\mathcal{R}' = (\mathbb{R}, +, \cdot, \text{Exp}, <, 0, 1)$, the real exponential field, remains o-minimal.
2000 - Macpherson, Marker, Steinhorn: *weakly o-minimal* structures
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Weakly o-minimal structures and Skolem functions
A *model* is a mathematical structure, with an ambient set ("universe") and a set of relation symbols, function symbols, and constant symbols ("language"), together with an intended meaning for them. The *theory* of a model $\mathcal{M}$ is the collection of first order sentences satisfied on $\mathcal{M}$. 
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A *model* is a mathematical structure, with an ambient set (“universe”) and a set of relation symbols, function symbols, and constant symbols (“language”), together with an intended meaning for them. The *theory* of a model $M$ is the collection of first order sentences satisfied on $M$.

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- $(\mathbb{R}, +, \cdot, <, 0, 1)$
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- $(\omega, +, \cdot, <, 0, 1)$- Arithmetical hierarchy
Definable sets

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- $(\mathbb{R}, +, \cdot, <, 0, 1)$- ?
A theory has **elimination of quantifiers** (QE) iff on any model $M$ and for any definable set $X \subseteq M$, there is a quantifier-free formula $\varphi(\bar{x})$ defining $X$. 
A theory has *elimination of quantifiers* (QE) iff on any model $\mathcal{M}$ and for any definable set $X \subseteq M$, there is a quantifier-free formula $\varphi(\bar{x})$ defining $X$.

Quantifier-free formulas can be written in *disjunctive normal form* (DNF):

$$
\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \varphi_{ij}
$$

for $\varphi_{ij}$ atomic or negated atomic.
The complex field is *strongly minimal*

The model $\mathcal{M} = (\mathbb{C}, +, \cdot, 0, 1)$ is QE. Using the DNF, one can show that the $\mathcal{M}$-definable subsets of $\mathbb{C}$ are all finite or cofinite.
The complex field is *strongly minimal*

- The model $\mathcal{M} = (\mathbb{C}, +, \cdot, 0, 1)$ is QE. Using the DNF, one can show that the $\mathcal{M}$-definable subsets of $\mathbb{C}$ are all finite or cofinite.

- Definable subsets of $\mathbb{C}^n$ are precisely the *constructible sets* (in the sense of Weyl).
Definition: o-minimal

An ordered structure \((M, <, \ldots)\) is o-minimal if every definable subset (with parameters) of \(M^1\) is a finite union of points and open intervals.
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- An o-minimal structure has an o-minimal theory [Knight-Pillay-Steinhorn, 1986].
Monotonicity & Cellular decomposition

**Monotonicity Theorem:** For every definable function $f : \mathcal{M} \to \mathcal{M}$, there is a partition of $\text{dom}(f)$ into finitely many intervals $I_1, \ldots, I_n$, such that for each $i \leq n$, $f \upharpoonright I_i$ is strictly monotone (strictly increasing, strictly decreasing, or constant).
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- **[Regular] cell decomposition**: Given a definable subset $Y \subseteq \mathcal{M}^n$, there is a finite partition $\mathcal{C}$ of $\mathcal{M}^n$ into [regular] cells such that $Y$ is a union of cells in $\mathcal{C}$.
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- For a cell $X$ in $\mathcal{M}^{n-1}$, a cell in $\mathcal{M}^n$ is the graph of a continuous $F : X \to \mathcal{M}$, or a difference function $(F, G)_X$. 

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Monotonicity & Cellular decomposition

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*[Regular] cell decomposition:* Given a definable subset $Y \subseteq \mathcal{M}^n$, there is a finite partition $\mathcal{C}$ of $\mathcal{M}^n$ into *regular* cells such that $Y$ is a union of cells in $\mathcal{C}$.

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For a cell $X$ in $\mathcal{M}^{n-1}$, a cell in $\mathcal{M}^n$ is the graph of a continuous $F : X \to \mathcal{M}$, or a difference function $(F, G)_X$.

Functions defined on regular cells are continuous and monotone in each variable.
Cellular decomposition: picture

\[(F, G)_X\]
Cellular decomposition: picture

\((F, G)_X\)
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\((F, G)_X\)
A complete picture of the definable sets in an o-minimal structure.
Consequences of cellular decomposition

- A complete picture of the definable sets in an o-minimal structure.
- Uniform finiteness
Consequences of cellular decomposition

- A complete picture of the definable sets in an o-minimal structure.
- Uniform finiteness $\iff$ o-minimal structures have o-minimal theories.

$(\mathcal{M}$ o-minimal and $\mathcal{M} \equiv \mathcal{N}$ implies $\mathcal{N}$ o-minimal)
Skolem functions (1)

Any o-minimal theory expanding a group has *definable Skolem functions*:
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O-minimal structures also have uniform elimination of imaginaries: every definable equivalence relation of $\mathcal{M}^n$ has a uniformly definable set of class representatives.

"Definable choice" [van den Dries, 1998]
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Skolem functions (2)

Example

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Skolem functions (2)

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Example

\[ M \]

\[ M^n \]
Skolem functions (2)

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Definition: Weakly o-minimal

An ordered structure \( (\mathcal{M}, <, \ldots) \) is weakly o-minimal if every definable subset of \( \mathcal{M}^1 \) is a finite union of convex sets. Initial observations:
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An ordered structure \((\mathcal{M}, <, \ldots)\) is \textit{weakly o-minimal} if every definable subset of \(\mathcal{M}^1\) is a finite union of \textit{convex sets}.

Initial observations:

- Convex sets do not have to have endpoints in the structure \(\mathcal{M}\).
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Definition: Weakly o-minimal

An ordered structure \((M, <, \ldots)\) is *weakly o-minimal* if every definable subset of \(M^1\) is a finite union of *convex sets*. Initial observations:

- Convex sets do not have to have endpoints in the structure \(M\).
- Any o-minimal structure is weakly o-minimal.
- Any weakly o-minimal structure which is Dedekind complete is also o-minimal.
Definition: Weakly o-minimal

An ordered structure \((M, <, \ldots)\) is weakly o-minimal if every definable subset of \(M^1\) is a finite union of convex sets.

Initial observations:

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- Any o-minimal structure is weakly o-minimal.
- Any weakly o-minimal structure which is Dedekind complete is also o-minimal.
- Thus, any weakly o-minimal structure with universe \(\mathbb{R}\) is o-minimal.
Adding a convex predicate

Since $\pi \not\in \mathbb{Q}$, the set $\{ x \in \mathbb{Q} : -\pi < x < \pi \}$ is convex in $\mathbb{Q}$, but not an interval.
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$$\mathcal{M}_1 = (\mathbb{Q}, <, +, P), \text{ where } P^{\mathcal{M}_1} = \{ x \in \mathbb{Q} : -\pi < x < \pi \}.$$
Since \( \pi \notin \mathbb{Q} \), the set \( \{ x \in \mathbb{Q} : -\pi < x < \pi \} \) is convex in \( \mathbb{Q} \), but not an interval.

\( M_1 = (\mathbb{Q}, <, +, P) \), where \( P^{M_1} = \{ x \in \mathbb{Q} : -\pi < x < \pi \} \).

\( M_2 = (\mathbb{R}^*, <, +, \cdot, U) \), where \( \mathbb{R}^* \) is a proper end extension of \( \mathbb{R} \), and \( U^{M_2} \) is the convex hull of \( \mathbb{R} \) in \( \mathbb{R}^* \).
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The most intuitive examples of weakly o-minimal structures. (Take a nice ordered structure, add a convex set!)
Adding a convex predicate

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- The most intuitive examples of weakly o-minimal structures. (Take a nice ordered structure, add a convex set!)
- \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are the two main paradigmatic examples of weakly o-minimal structures.
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Monotonicity Theorem [Arefiev, 1997] For $\mathcal{M}$ weakly o-minimal, for every definable function $f : \mathcal{M} \to \mathcal{M}$, there is a partition of $\text{dom}(f)$ into finitely many convex sets $U_1, \ldots, U_n$, such that for each $i \leq n$, $f \restriction U_i$ is locally strictly monotone (strictly increasing, strictly decreasing, or constant).
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$\mathcal{M}$ has the Finitary Monotonicity Property (FMP) if $f \upharpoonright U_i$ with $f$ and $U_i$ as above is strictly monotone.
Weakly o-minimal structures

- **Monotonicity Theorem** [Arefiev, 1997] For $\mathcal{M}$ weakly o-minimal, for every definable function $f : \mathcal{M} \rightarrow \mathcal{M}$, there is a partition of $\text{dom}(f)$ into finitely many *convex sets* $U_1, \ldots, U_n$, such that for each $i \leq n$, $f \upharpoonright U_i$ is *locally* strictly monotone (strictly increasing, strictly decreasing, or constant).

- $\mathcal{M}$ has the *Finitary Monotonicity Property* (FMP) if $f \upharpoonright U_i$ with $f$ and $U_i$ as above is strictly monotone.

- FMP is not true of general weakly o-minimal structures.
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A structure without “reasonable” monotonicity

Why is there no contradiction to weak o-minimality?
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Given that there is no hope for the above in the general case, look only at weakly o-minimal structures expanding a group. What can we say?
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General facts

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Weakly o-minimal structures and Skolem functions
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- Any definable subgroup is convex $\Rightarrow$ weakly o-minimal groups satisfy DOAG.
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Valuational: definition and examples

A cut $\langle C, D \rangle$ of $\mathcal{M}$
A cut $\langle C, D \rangle$ of $\mathcal{M}$ is \textit{valuational} if there is $\varepsilon > 0$ such that $C + \varepsilon = C$. 

\begin{center}
\begin{tikzpicture}
    \draw[->] (0,0) -- (5,0) node[right] {$D$};
    \draw[->] (0,0) -- (-5,0) node[left] {$\mathcal{M}$};
    \fill (2,0) circle (2pt) node[above] {$C$};
    \fill (-2,0) circle (2pt) node[above] {$0$};
    \fill (0,-0.5) circle (2pt) node[below] {$\varepsilon$};
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Diagram: 

```
0  ε  C  g  D  M
```

A weakly o-minimal group $\mathcal{M}$ is valuational if $\mathcal{M}$ has a definable valuational cut; equivalently, $\mathcal{M}$ has a definable proper subgroup. 

By the subgroup characterization, the end-extension of $\mathbb{R}$ is valuational. Using the other characterization, the rationals with convex predicate for $\pi$ is nonvaluational.
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![Diagram of a cut](image)

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In the meantime

Before discovering the work of Wencel, we found a partial result:

- If $Th(\mathcal{M})$ is weakly o-minimal and valuational, then $\mathcal{M}$ has finitary monotonicity.
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- If $Th(\mathcal{M})$ is weakly o-minimal and valuational, then $\mathcal{M}$ has finitary monotonicity.
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- If $Th(\mathcal{M})$ has definable Skolem functions and elimination of imaginaries, then there is no convex definable equivalence relation with infinitely many infinite classes $\Rightarrow \mathcal{M}$ has finitary monotonicity.
- This also means $Th(\mathcal{M})$ does not have a proper definable subgroup.
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Valuational structures

There is no hope of a "nice" cellular decomposition. In \((\mathbb{R}^*, +, \times, U)\), define \(X \subseteq \mathbb{R}^*\) by the formula \(U(y - x)\).
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Basic example

In a weakly o-minimal structure, we have definable sets without endpoints in the structure.

Hard to find the midpoint of a convex set without endpoints.

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Dashing hopes

Example: \((\mathbb{Q}, +, -, <, P, \lambda_q)_{q \in \mathbb{Q}}\).

Has Q.E., so we can say what the definable functions are. Every definable function is piecewise linear.

\[ \phi(x, y) = P(x) \land P(y) \land x < y. \]

\[ F: \mathbb{Q} \to \mathbb{Q} \text{ such that } x < f(x) < \pi. \]

The result generalizes to any nonvaluational cut added to an o-minimal group.
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**Example:** $(\mathbb{Q}, +, -, <, P, \lambda_q)_{q \in \mathbb{Q}}$. Has Q.E., so we can say what the definable functions are.
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- The result generalizes to any nonvaluational cut added to an o-minimal group.
Generalizing the nonvaluational result (1)

**Theorem:** $(\mathcal{M}, +, <, \ldots)$ o-minimal, and $U$ a new nonvaluational convex subset. Then $(\mathcal{M}, U)$ does not have definable Skolem functions.

Proof (Outline):

- $\mathcal{M}$ o-minimal, $U$ a new convex subset. Define $tp(\sup U/\mathcal{M})$ the type in $\mathcal{M}$ of the supremum of $U$.
- Given $\mathcal{M}$ as above, let $b$ realize $tp(\sup U/\mathcal{M})$.
- O-minimal structures have prime models over sets; let $\mathcal{N} = \text{pr}(\mathcal{M} \cup \{b\})$.

**Lemma:** $\mathcal{M}$ is dense in $\mathcal{N}$ if and only if $U$ is nonvaluational.

**Theorem** [van den Dries, 1998]: Every function $F: \mathcal{M} \to \mathcal{M}$ definable in $(\mathcal{N}, \mathcal{M})$ is piecewise definable in $\mathcal{M}$.
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- Given $\mathcal{M}$ as above, let $b$ realize $tp(\text{sup}U/M)$.
Theorem: \((M, +, <, \ldots)\) o-minimal, and \(U\) a new nonvaluational convex subset. Then \((M, U)\) does not have definable Skolem functions.

Proof (Outline):

- \(M\) o-minimal, \(U\) a new convex subset. Define \(tp(\text{sup}U/M)\) = the type in \(M\) of the supremum of \(U\).
- Given \(M\) as above, let \(b\) realize \(tp(\text{sup}U/M)\).
- O-minimal structures have prime models over sets; let \(N = pr(M \cup \{b\})\).
Generalizing the nonvaluational result (1)

**Theorem:** $(\mathcal{M}, +, <, \ldots)$ o-minimal, and $U$ a new nonvaluational convex subset. Then $(\mathcal{M}, U)$ does not have definable Skolem functions.

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- $\mathcal{M}$ o-minimal, $U$ a new convex subset. Define $tp(\sup U/M) = \text{the type in } \mathcal{M} \text{ of the supremum of } U$.
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Proof (Outline):

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3. O-minimal structures have prime models over sets; let \(\mathcal{N} = pr(\mathcal{M} \cup \{b\})\).
4. **Lemma:** \(\mathcal{M}\) is dense in \(\mathcal{N}\) if and only if \(U\) is nonvaluational.
5. **Theorem** [van den Dries, 1998]: Every function \(F : \mathcal{M} \rightarrow \mathcal{M}\) definable in \((\mathcal{N}, \mathcal{M})\) is piecewise definable in \(\mathcal{M}\).
Generalizing the nonvaluational result (2)

Translate the dense pair result:

- Any $F$ definable in $(\mathcal{M}, U)$ is definable in $(\mathcal{N}, \mathcal{M})$: replace $U(x)$ with $x < b$. 
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We may colloquially say $\lim_{x \rightarrow \sup(U)} (F(x) - x) = 0$

...$
\ldots \mathcal{M} \models \theta = (\forall \varepsilon > 0)(\forall x \in U)(\exists \delta > 0)(x + \delta \in U \land 0 < F(x + \delta) - x < \varepsilon)$
Generalizing the nonvaluational result (3)

\[ F \subseteq (\mathbb{N}, \mathbb{M}) \]

By the dense pair result, there is an interval \( I \) overlapping with \( \sup(U) \) and a partial function \( F' \) definable in \( \mathbb{M} \) such that \( F \upharpoonright U = F' \).

\( U \) is nonvaluational, so given \( \delta > 0 \), there is \( a \in U \) such that \( a + \delta > U \).

This contradicts \( \mathbb{M} \models \theta \).
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By the dense pair result, there is an *interval* $I$ overlapping with $\text{sup}(U)$ and a partial function $F'$ definable in $M$ such that $F|U = F'$. 

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Generalizing the nonvaluational result (3)

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- This contradicts $\mathcal{M} \models \theta$. 
1 Background
   - Brief History
   - Model - definable - QE - DNF - o-minimal - Skolem function
   - Weakly o-minimal structures
   - Monotonicity

2 Valuational and nonvaluational cuts
   - Motivation
   - Definable subgroups
   - Good news
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3 Skolem functions in valuational structures
   - $T$-immunity
   - Technique for elimination
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Definition of $T$-immunity
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Fix $T$, an o-minimal expansion of a group; let $\mathcal{M} \models T$. A subset $V \subset M$ is $T$-immune if $V$ is convex, and for any 0-definable continuous partial function $F : \mathcal{M} \to \mathcal{M}$, $F(V) \subseteq V$. We say $(\mathcal{M}, V) \models T_{\text{immune}}$. 
For certain valuational structures, we can perform a direct calculation of quantifier elimination.

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Example: If $\mathcal{L} = \{+, <, 0\}$, $\mathcal{M}$ is any o-minimal group, and $V$ any proper convex subgroup, then $(\mathcal{M}, V)$ is $T$-immune.
The main result

**Theorem:** Let $\mathcal{M} \models T$ be an o-minimal expansion of a group which admits elimination of quantifiers, and $V$ a $T$-immune set. Let $\varepsilon, c$ be new constant symbols, and $\varepsilon^\mathcal{M} \in V$ and $c^\mathcal{M}$ a positive element of $\mathcal{M} \setminus V$. Then $Th((\mathcal{M}, V)_c)$ admits elimination of quantifiers and has definable Skolem functions.
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- To eliminate quantifiers directly, it suffices to eliminate a single existential quantifier from a primitive formula (finite conjunction of atomic and negatomic formulas).
- This comes from the Disjunctive Normal Form theorem.
- To prove QE this way often uses induction on formula complexity.
- We proceed on a more intuitive path.
Lemma: Let $U_0, \ldots, U_{n-1}$ be convex subsets of a totally ordered set.
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**Lemma:** Let $U_0, \ldots, U_{n-1}$ be convex subsets of a totally ordered set. Then there are $j, k < n$ (possibly $j = k$) such that

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Lemma: Let $U_0, \ldots, U_{n-1}$ be convex subsets of a totally ordered set. Then there are $j, k < n$ (possibly $j = k$) such that
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In particular, $\bigcap_{i<n} U_i$ is nonempty if and only if for each $i, j < n$, $U_j \cap U_k$ is nonempty.
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Consequence of the lemma:

1. Given a primitive formula \( \exists x \bigwedge_i \varphi_i(x, \bar{y}) \)
2. Suppose \((\ast)\): \(\varphi_i(M, \bar{b})\) is convex for each \(\bar{b} \in M^n\)
3. Then \(\exists x \left( \bigwedge_i \varphi_i(x, \bar{y}) \right) \iff \bigwedge_{i<j} \exists x (\varphi_i(x, \bar{y}) \land \varphi_j(x, \bar{y})) \)
Consequence of the lemma:

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- $M$ is o-minimal and QE! We may assume every formula $\varphi(x, \bar{y})$ of $\mathcal{L}$ satisfies $(\ast)$. 
To get the convexity of \( \varphi(x, \bar{b}) \) for formulas of the expanded language, we form an extension by definitions:

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- Replace $\neg V$ with $L \lor R$.
- “Reorganize.”
In the expanded language we have to deal with formulas of the form $\exists x V(F(x, \bar{y}))$. 
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- $\exists x V(F(x, \bar{y}))$ becomes $\exists x \bigvee_i (V(F_i(x, \bar{y})) \land x\bar{y} \in C_i)$
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- Each regular cell $C_i$ is quantifier-free definable, since $\mathcal{M}$ eliminates quantifiers.
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- Each regular cell $C_i$ is quantifier-free definable, since $M$ eliminates quantifiers.
- Each $F(C_{i\bar{b}})$ is convex, since $F$ is monotone in $x$. 

$\text{QE}$
Define a $\ast$-atomic formula (note that each is convex and quantifier-free):
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- $\varphi(x, \bar{y})$ for $\varphi$ an $\mathcal{L}$-formula with $\varphi(M, \bar{b})$ convex for each $\bar{b} \in M^n$
- $V(F(x, \bar{y}))$,
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Define a $\star$-atomic formula (note that each is convex and quantifier-free):

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It now suffices to eliminate the existential from a “$\tilde{*}$-primitive formula.”
Define a \( \star \)-atomic formula (note that each is convex and quantifier-free):

- \( \varphi(x, \bar{y}) \) for \( \varphi \) an \( \mathcal{L} \)-formula with \( \varphi(\mathcal{M}, \bar{b}) \) convex for each \( \bar{b} \in \mathcal{M}^n \)
- \( V(F(x, \bar{y})) \),
- \( L(F(x, \bar{y})) \),
- \( R(F(x, \bar{y})) \), for a 0-definable partial function \( F \) on a regular cell

Get a \( \star \)-disjunctive normal form theorem (\( \star \)-DNF).

It now suffices to eliminate the existential from a “\( \star \)-primitive formula.”

Using the topological lemma, it suffices to eliminate the existential from a pair of \( \star \)-atomic formulas.
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- Look at a $\ast$-primitive formula $\varphi(\bar{x}, y) = \bigwedge_i \varphi_i(\bar{x}, y)$. 
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- Look at a $*$-primitive formula $\varphi(\bar{x}, y) = \bigwedge_i \varphi_i(\bar{x}, y)$.
- By the lemma, given $\bar{a}$ from $\mathbb{R}^*$,
  $\varphi(\bar{a}, \mathcal{M}) = (\varphi_j(\bar{a}, y) \land \varphi_k(\bar{a}, y))^{\mathcal{M}}$ for some $j, k$. 

Weakly o-minimal structures and Skolem functions
Skolem functions

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- If we have a Skolem function for each pairwise conjunction of $\ast$-atomic formulas, we have a function for $\varphi$. 
Again, use the topological lemma:

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- If we have a Skolem function for each pairwise conjunction of \(*\)-atomic formulas, we have a function for \(\varphi\).

- For an arbitrary formula, write in \(*\)-DNF, and use the functions for each primitive formula.
Again, use the topological lemma:

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Again, use the topological lemma:

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- Do this simultaneously with quantifier elimination.
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- $T$-immunity
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Assume \((\mathcal{M}, V)\) is \(T\)-immune.
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- If \(F : \mathcal{M}^n \to \mathcal{M}\) is partial, 0-definable, continuous, then
  \(F(V^n) \subseteq V\).
Technical lemmas ("Math")

Assume \((\mathcal{M}, V)\) is \(T\)-immune.

- If \(F : \mathcal{M}^n \to \mathcal{M}\) is partial, 0-definable, continuous, then \(F(V^n) \subseteq V\).
- "\(V\)-faithfulness:" Given \(F(x, \bar{y}), \bar{b} \in \mathcal{M}^n\) fixed, for any \(a \in \mathcal{M}\), if \(F(a, \bar{b}) \in V\), then for any \(e \in V\), \(F(a + e \in V)\).
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- "No slow-growing functions:" If \(F_{\bar{b}}\) is strictly increasing and \(F(a, \bar{b}) \in \mathcal{V}\), then for any \(d \in \mathcal{M} \setminus \mathcal{V}\), \(F(a + d, \bar{b}) \notin \mathcal{V}\).
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- "V-faithfulness:" Given \(F(x, \bar{y}), \bar{b} \in \mathcal{M}^n\) fixed, for any \(a \in \mathcal{M}\), if \(F(a, \bar{b}) \in V\), then for any \(e \in V\), \(F(a + e \in V)\).

- "No slow-growing functions:" If \(F_{\bar{b}}\) is strictly increasing and \(F(a, \bar{b}) \in V\), then for any \(d \in \mathcal{M}\setminus V\), \(F(a + d, \bar{b}) \notin V\).

- "If \(F\) and \(G\) are far apart, they stay far apart:" If \(F_{\bar{b}}\) and \(G_{\bar{b}}\) are both strictly increasing, and there is \(a\) such that \(F(a, \bar{b}) \in V\) and \(G(a, \bar{b}) \notin V\), then for every \(a\), \(F(a, \bar{b}) \in V \Rightarrow G(a, \bar{b}) \notin V\).
Case analysis (1)

We may assume a $*$-primitive formula takes one of the following forms:

1. $\exists x \, \varphi(x, \bar{y})$, for $\varphi$ an $L$-formula and $\varphi(M, \bar{b})$ convex for every $\bar{b} \in M$
2. $\exists x \, (V(F(x, \bar{y})))$
3. $\exists x \, (L(F(x, \bar{y})))$ or $\exists x \, (R(F(x, \bar{y})))$
4. $\exists x \, (\varphi(x, \bar{y}) \land V(F(x, \bar{y})))$ or $\exists x \, (\varphi(x, \bar{y}) \land L(G(x, \bar{y})))$ or $\exists x \, (\varphi(x, \bar{y}) \land R(G(x, \bar{y})))$ or $\exists x \, (\varphi(x, \bar{y}) \land R(G(x, \bar{y})))$
5. $\exists x \, (V(F(x, \bar{y})) \land V(G(x, \bar{y})))$ or $\exists x \, (V(F(x, \bar{y})) \land L(G(x, \bar{y})))$ or $\exists x \, (V(F(x, \bar{y})) \land R(G(x, \bar{y})))$ or $\exists x \, (V(F(x, \bar{y})) \land R(G(x, \bar{y})))$
6. $\exists x \, (L(F(x, \bar{y})) \land L(G(x, \bar{y})))$ or $\exists x \, (L(F(x, \bar{y})) \land R(G(x, \bar{y})))$ or $\exists x \, (R(F(x, \bar{y})) \land R(G(x, \bar{y})))$
Case analysis (1)

We may assume a $\ast$-primitive formula takes one of the following forms:

(1) $\exists x \varphi(x, \bar{y})$, for $\varphi$ an $\mathcal{L}$-formula and $\varphi(\mathcal{M}, \bar{b})$ convex for every $\bar{b} \in \mathcal{M}^n$
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2. $\exists x \left( V(F(x, \bar{y})) \right)$
3. $\exists x \left( L(F(x, \bar{y})) \right)$ or $\exists x \left( R(F(x, \bar{y})) \right)$
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1. \( \exists x \varphi(x, \bar{y}) \), for \( \varphi \) an \( \mathcal{L} \)-formula and \( \varphi(M, \bar{b}) \) convex for every \( \bar{b} \in M^n \)
2. \( \exists x (V(F(x, \bar{y}))) \)
3. \( \exists x (L(F(x, \bar{y}))) \) or \( \exists x (R(F(x, \bar{y}))) \)
4. \( \exists x (\varphi(x, \bar{y}) \land V(F(x, \bar{y}))) \) or \( \exists x (\varphi(x, \bar{y}) \land L(F(x, \bar{y}))) \) or \( \exists x (\varphi(x, \bar{y}) \land R(F(x, \bar{y}))) \), for \( \varphi \) as above
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5. $\exists x \left( V(F(x, \bar{y})) \land V(G(x, \bar{y})) \right)$
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2. \( \exists x \left( V(F(x, \bar{y})) \right) \)

3. \( \exists x \left( L(F(x, \bar{y})) \right) \) or \( \exists x \left( R(F(x, \bar{y})) \right) \)

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5. \( \exists x \left( V(F(x, \bar{y})) \land V(G(x, \bar{y})) \right) \)

6. \( \exists x \left( V(F(x, \bar{y})) \land L(G(x, \bar{y})) \right) \) or \( \exists x \left( V(F(x, \bar{y})) \land R(G(x, \bar{y})) \right) \) or \( \exists x \left( L(F(x, \bar{y})) \land L(G(x, \bar{y})) \right) \) or \( \exists x \left( L(F(x, \bar{y})) \land R(G(x, \bar{y})) \right) \) or \( \exists x \left( R(F(x, \bar{y})) \land R(G(x, \bar{y})) \right) \)
Case analysis (2)

(1) For \( \mathcal{L} \)-formulas, QE and Skolem functions are known.
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(2) For \( V(F(x, \vec{y})) \): fix \( \vec{b} \in M^n \); assume the domain of \( F_{\vec{b}} \) is a bounded interval \((\alpha, \beta)\), and \( F_{\vec{b}} \) increasing.
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$L(F(\alpha + \varepsilon, \bar{b})) \land R(F(\beta - \varepsilon, \bar{b}))$, or
Case analysis (2)

1. For $\mathcal{L}$-formulas, QE and Skolem functions are known.
2. For $V(F(x, \bar{y}))$: fix $\bar{b} \in M^n$; assume the domain of $F_{\bar{b}}$ is a bounded interval $(\alpha, \beta)$, and $F_{\bar{b}}$ increasing. Check endpoint behavior at $\alpha + \varepsilon$ and $\beta - \varepsilon$.

$$V(F(\alpha + \varepsilon, \bar{b})), \text{ or } V(F(\beta - \varepsilon, \bar{b})),$$
Case analysis (2)

(1) For $\mathcal{L}$-formulas, QE and Skolem functions are known.

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\[ V(F(\alpha + \varepsilon, \bar{b})) \land R(F(\beta - \varepsilon, \bar{b})), \text{ or } V(F(\beta - \varepsilon, \bar{b})), \text{ or } V(F(\alpha + \varepsilon, \bar{b})). \]
(3) For $R(F(x, \bar{y}))$

(3) For $R(F(\alpha + \varepsilon, \bar{b}))$ or $R(F(\beta - \varepsilon, \bar{b}))$. Other cases similar.
Case analysis (3)

(3) For $R(F(x, \bar{y}))$: fix $\bar{b} \in \mathcal{M}^n$; again assume the domain of $F_{\bar{b}}$ is a bounded interval $(\alpha, \beta)$, and $F_{\bar{b}}$ increasing.
Case analysis (3)

(3) For $R(F(x, \bar{y}))$: fix $\bar{b} \in \mathcal{M}^n$; again assume the domain of $F_{\bar{b}}$ is a bounded interval $(\alpha, \beta)$, and $F_{\bar{b}}$ increasing. Check endpoint behavior at $\alpha + \varepsilon$ and $\beta - \varepsilon$. 
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\[ R(F(\beta - \varepsilon, \bar{b})) \text{, or} \]

\[ R(F(\alpha + \varepsilon, \bar{b})) \text{, or} \]
(3) For \( R(F(x, y)) \): fix \( \bar{b} \in \mathcal{M}^n \); again assume the domain of \( F_{\bar{b}} \) is a bounded interval \((\alpha, \beta)\), and \( F_{\bar{b}} \) increasing. Check endpoint behavior at \( \alpha + \varepsilon \) and \( \beta - \varepsilon \).

\[ R(F(\beta - \varepsilon, \bar{b})), \text{ or} \]

\[ R(F(\alpha + \varepsilon, \bar{b})). \]
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Case analysis (3)

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$\bullet$ $R(F(\alpha + \varepsilon, \bar{b}))$. 

03/10/10 - Christopher Shaw
Case analysis (3)

- Other cases similar.
1 Background
   - Brief History
   - Model - definable - QE - DNF - o-minimal - Skolem function
   - Weakly o-minimal structures
   - Monotonicity

2 Valuational and nonvaluational cuts
   - Motivation
   - Definable subgroups
   - Good news
   - Pathologies

3 Skolem functions in valuational structures
   - $T$-immunity
   - Technique for elimination
   - Corollaries
Corollaries

Let \((\mathcal{M}, +, <, \ldots)\) be o-minimal and \(U\) a convex subset of \(\mathcal{M}\).
Corollaries

Let $(\mathcal{M}, +, <, \ldots)$ be o-minimal and $U$ a convex subset of $\mathcal{M}$.

- If $(\mathcal{M}, U)$ is nonvaluational, there are no Skolem functions.
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Let $(\mathcal{M}, +, <, \ldots)$ be o-minimal and $U$ a convex subset of $\mathcal{M}$.

- If $(\mathcal{M}, U)$ is nonvaluational, there are no Skolem functions.
- If $(\mathcal{M}, U)$ is valuational, $(\mathcal{M}, U)$ has a convex definable subgroup.
Corollaries

Let \((M, +, <, \ldots)\) be o-minimal and \(U\) a convex subset of \(M\).

- If \((M, U)\) is nonvaluational, there are no Skolem functions.
- If \((M, U)\) is valuational, \((M, U)\) has a convex definable subgroup, thus cannot have Skolem functions and elimination of imaginaries.
Corollaries

Let \((\mathcal{M}, +, <, \ldots)\) be o-minimal and \(U\) a convex subset of \(\mathcal{M}\).

- If \((\mathcal{M}, U)\) is nonvaluational, there are no Skolem functions.
- If \((\mathcal{M}, U)\) is valuational, \((\mathcal{M}, U)\) has a convex definable subgroup, thus cannot have Skolem functions and elimination of imaginaries. Therefore if \((\mathcal{M}, U)\) has Skolem functions and elimination of imaginaries, \(U\) must be an interval, and \((\mathcal{M}, U)\) is o-minimal.
Let \((\mathcal{M}, +, <, \ldots)\) be o-minimal and \(U\) a convex subset of \(\mathcal{M}\).

- If \((\mathcal{M}, U)\) is nonvaluational, there are no Skolem functions.
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  Therefore if \((\mathcal{M}, U)\) has Skolem functions and elimination of imaginaries, \(U\) must be an interval, and \((\mathcal{M}, U)\) is o-minimal.

- If \((\mathcal{M}, U)\) is \(T\)-immune, then \((\mathcal{M}, U)_c\) has Skolem functions, but still has a definable proper subgroup.
Let \((M, +, <, \ldots)\) be o-minimal and \(U\) a convex subset of \(M\).

- If \((M, U)\) is nonvaluational, there are no Skolem functions.
- If \((M, U)\) is valuational, \((M, U)\) has a convex definable subgroup, thus cannot have Skolem functions and elimination of imaginaries.

Therefore if \((M, U)\) has Skolem functions and elimination of imaginaries, \(U\) must be an interval, and \((M, U)\) is o-minimal.

- If \((M, U)\) is \(T\)-immune, then \((M, U)_c\) has Skolem functions, but still has a definable proper subgroup, thus \((M, U)\) does not eliminate imaginaries.
Future work - generalizing the results
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- Generalize Skolem function technique to “$T$-convex” structures (van den Dries)
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- $T$-convexity is *sufficient*; is it necessary?
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- $T$-convexity is *sufficient*; is it necessary? (modulo trivialities)
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- Generalize Skolem function technique to “$T$-convex” structures (van den Dries)
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- Speculation:
Future work - generalizing the results

- Generalize Skolem function technique to “$T$-convex” structures (van den Dries)
- $T$-convexity is *sufficient*; is it necessary? (modulo trivialities)
- Speculation: yes.