Definability in weakly o-minimal structures

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Outline

1. Background and definitions
2. o-minimal structures (the “classical” results)
3. Weakly o-minimal structures
4. Hopes, dreams, and methods

Model theory: Background

Model theoretic programme: Study sets *definable* in a *model*.

- For our purposes, "definable" $\leftrightarrow$ described by a *formula* φ using symbols from a fixed language L. (L is always assumed to contain logical connectives $\forall$, $\exists$, $\vee$, $\wedge$, $\neg$ and $=$.)

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- Ask: how complex can these definable sets be?

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Can a computer prove all of the theorems of T?

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Definable sets of M_1^1: Not many! All definable sets are finite or cofinite - as simple as possible:

$\varphi(x) := x = 7 \vee x = 15 \vee x = \pi$

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Example: $\mathcal{M}_2 = (\mathbb{Q}, <)$

Definable sets of M_2^1*:* Again, not many! Just finite unions of intervals:

$\psi(x) := x < 5 \wedge x > 2$

Solution set is simply the rational interval $(2, 5)$

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- Simplest definable subsets called *recursive*.
- Collection of definable sets arranged into the *arithmetic hierarchy*: still countable, but unwieldy.
- Consequence: if one can ‘interpret’ $\mathcal{N}$ in another structure $\mathcal{N}'$, then $\mathcal{N}'$ is just as complicated.

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- *Weakly o-minimal model* $\mathcal{M}$: All definable subsets of M^1 are finite unions of convex sets (not necessarily intervals). (Slang: A formula φ only “changes its mind” finitely many times on $\mathcal{M}$.)

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- Note: This means that adding significant complexity to the language, in particular allowing polynomials of arbitrary degree, adds no complexity to the definable subsets of M.
- May discuss a *strongly minimal theory*, such that every model of the theory is also strongly minimal.

o-minimal models and theories

In any infinite linear order without endpoints, strong minimality is *not* possible (as witnessed by the formula $x > a$ for any a). Hence, whenever a linear ordering is present in the model, o-minimal is the simplest possible case.

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- Example: $\mathbb{Q}$ considered as a divisible ordered abelian group, $(\mathbb{Q}, +, <, 0)$, *is* o-minimal (also via quantifier elimination)...
- ... but, $(\mathbb{Q}, +, \cdot, <, 0, 1)$, the ordered field, is *not* o-minimal. In fact, this is the *worst* possible case, as one can define the integers in this structure with a single formula (Robinson 1949).

o-minimal: key tools

The main tool in the study of o-minimal structures is the Monotonicity Theorem (Pillay, Steinhorn 1986):

[Consider all linearly ordered models to be densely ordered.] Suppose $\mathcal{M}$ is o-minimal, and let $f : M \to M$ be a definable function. Then there is a definable finite partition of dom$f$ into intervals $\{U_i : i \leq n\}$ such that for each i, $f|U_i$ is *strictly monotone* and *continuous*.

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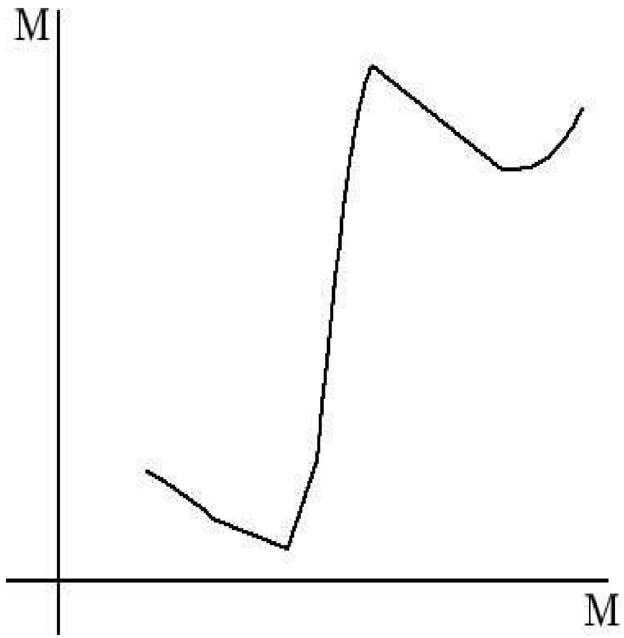

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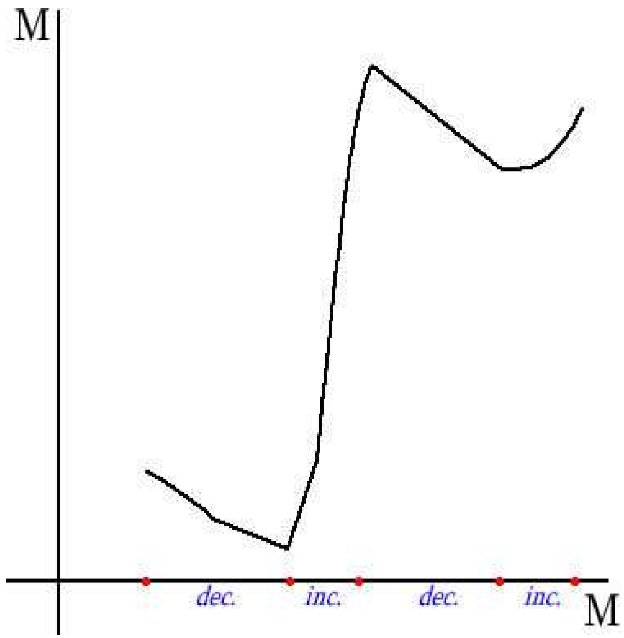

o-minimal: key tools (II)

Cellular decomposition: (Knight, Pillay, Steinhorn 1986) The idea: Given an o-minimal structure $\mathcal{M}$, every definable subset of M^n can be broken down into finitely many *cells*, where a cell can be thought of as a "nice" subset of M^n.

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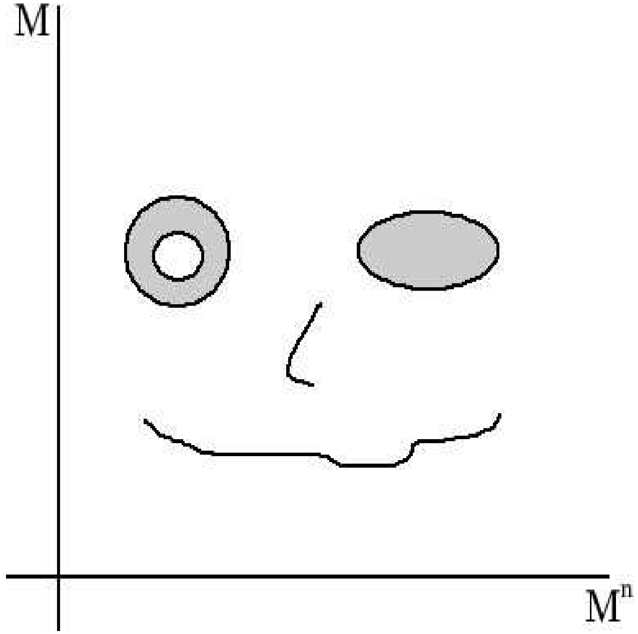

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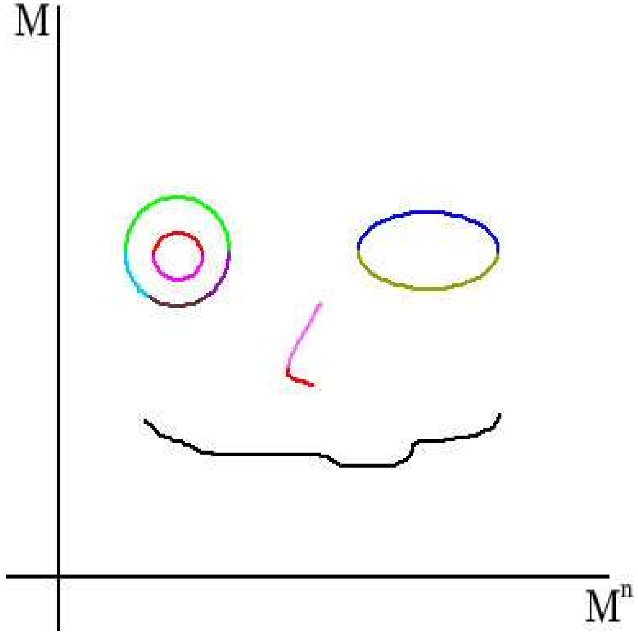

o-minimal: main results

Theorem: (Knight, Pillay, Steinhorn 1986) Let $\mathcal{M}$ be o-minimal, and $\mathcal{N} \equiv \mathcal{M}$. Then $\mathcal{N}$ is o-minimal. (Uses cell decomposition): “Every o-minimal model has an *o-minimal theory*.”

- From this fact, the study of o-minimal *models* becomes the study of o-minimal *theories*. Once you know a theory is o-minimal, *cellular decomposition* shows that any such structure is quite manageable in terms of definable sets.

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- Big Question: Which structures *have* o-minimal theories?
- Subquestion: $(\mathbb{R}, +, \cdot, <, 0, 1)$, the real field, is o-minimal. Which expansions (by language) of the real field are o-minimal?

Weakly o-minimal structures

Example: Take $(\mathbb{Q}, <)$, the rational line, and add a unary predicate symbol P to the language. Let $\mathcal{M}$ be $(\mathbb{Q}, <, P)$, where $P^{\mathcal{M}}$ is intepreted as $\{q \in \mathbb{Q} : -\pi < q < \pi\}$, or $(-\pi, \pi) \cap \mathbb{Q}$.

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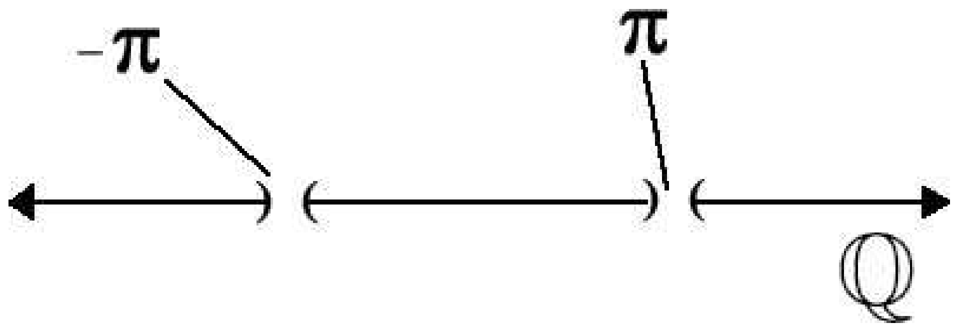

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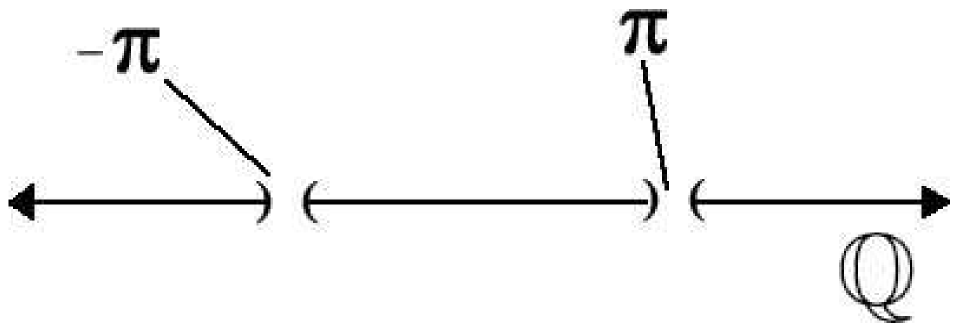

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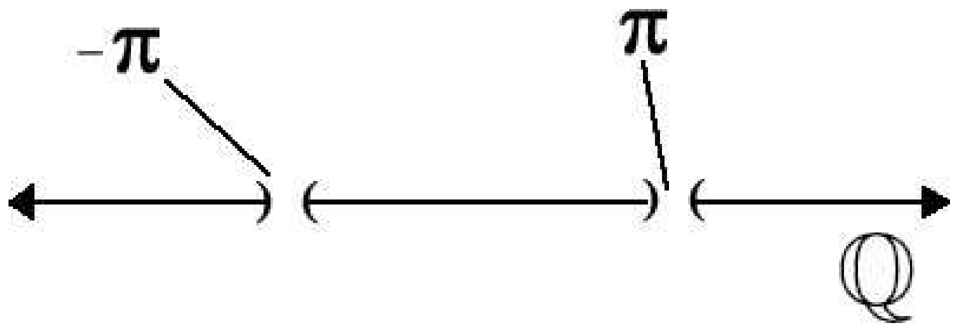

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By a theorem of Poizat, this structure is weakly o-minimal.

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- Which structures *are* weakly o-minimal?
- Given an o-minimal structure, which of its expansions by language are *weakly* o-minimal?

Weakly o-minimal structures: Failures

Investigate the possible analogues of o-minimal theories to weakly o-minimal structures:

- Monotonicity fails.

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- Monotonicity fails.
- Thus, cell decomposition fails.
- There exist $\mathcal{M} \equiv \mathcal{N}$ with $\mathcal{M}$ weakly o-minimal, and $\mathcal{N}$ *not* weakly o-minimal. Hence not every weakly o-minimal structure has a weakly o-minimal theory.

Weak monotonicity

A failure of monotonicity:
Let $\mathcal{M} = (\mathbb{Q}_L + (\mathbb{Q} \times \mathbb{Q})_R, <, f)$, where $<$ mimics the lexicographic order on $(\mathbb{Q} \times \mathbb{Q})_R$, and $f : (\mathbb{Q} \times \mathbb{Q})_R \longrightarrow \mathbb{Q}_L$ such that $f((p_R, q_R)) = p_L$.

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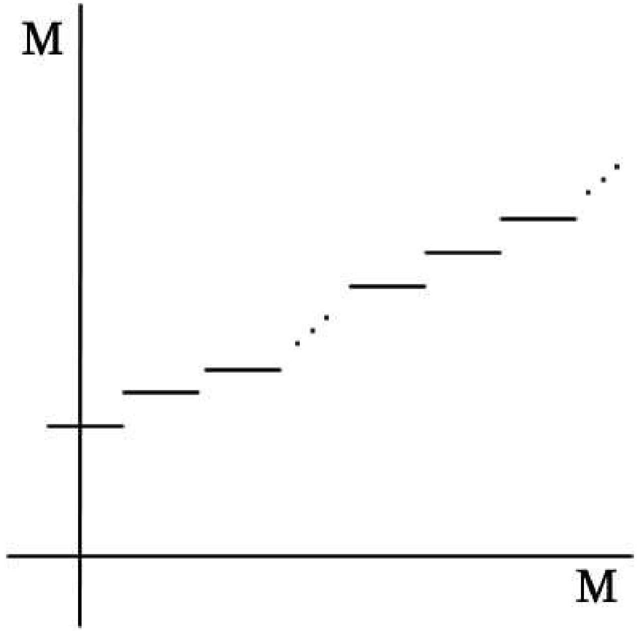

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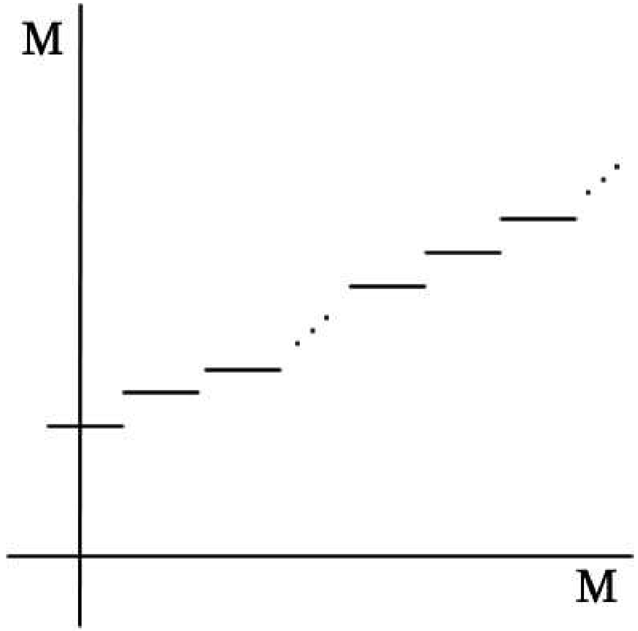

Note that in this case, f is still *locally* constant everywhere, but not globally constant.

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- Weak monotonicity result leads to a version of cell decomposition without continuous boundary functions.
- Still not very useful in establishing tameness of definable sets in higher dimensions.

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- Definition: A weakly o-minimal structure $\mathcal{M}$ *has strong monotonicity* iff for every definable function $f : M \longrightarrow M$, there is a definable finite partition of $dom(f)$ into convex sets U_i such that for each i, $f|U_i$ is strictly monotone and continuous.

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- *Strong monotonicity* is the best possible analogue for the monotonicity found in o-minimal structures.
- Big question: When do we have strong monotonicity?

A New Approach

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- Ordered structures do not have simple theories.
- Is there a parallel concept for ordered structures?

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- The concept is still being tested.

Current Results

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- Note: the weaker version of cell decomposition does not guarantee a weakly o-minimal theory.

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- Ask: is there a slight weakening of strong monotonicity that will allow locally constant functions, but still achieve strong monotonicty of decreasing and increasing functions?
- Does rosiness guarantee strong monotonicity?