Definability in weakly o-minimal structures

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Outline

1. Background and definitions
2. o-minimal structures (the “classical” results)
3. Weakly o-minimal structures
4. Hopes, dreams, and methods
Model theory: Background

Model theoretic programme: Study sets *definable* in a *model.*

For our purposes, “definable” $\iff$ described by a *formula* $\varphi$ using symbols from a fixed language $L$. ($L$ is always assumed to contain logical connectives $\forall, \exists, \lor, \land, \neg$ and $=$.)
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Definability in weakly o-minimal structures – p.3/36
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- $Th(\mathcal{M})$ is the collection of \textit{sentences} (formulas with no free variables) satisfied by $\mathcal{M}$. $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$, or $\mathcal{M} \equiv \mathcal{N}$, iff $Th(\mathcal{M}) = Th(\mathcal{N})$. 

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- Ask: how complex can these definable sets be?
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Model Theory: Background (II)

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Question is equivalent to whether $T \models \exists x \exists y \varphi(x, y)$, where

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Can a computer prove all of the theorems of $T$?
Example: $\mathcal{M}_1 = (\mathbb{C})$, no non-logical symbols

*Definable sets of $\mathcal{M}_1^1$: Not many! All definable sets are finite or cofinite - as simple as possible:*

$\varphi(x) := x = 7 \lor x = 15 \lor x = \pi$

Solution set is $\{\pi, 7, 15\}$
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Example: $\mathcal{M}_2 = (\mathbb{Q}, <)$

*Definable sets of $\mathcal{M}_2^1$*: Again, not many! Just finite unions of intervals:

$\psi(x) := x < 5 \land x > 2$

Solution set is simply the rational interval $(2, 5)$
Example: $\mathcal{N} = (\mathbb{N}, +, \cdot, <, 0, 1)$

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- Simplest definable subsets called *recursive*.
- Collection of definable sets arranged into the *arithmetic hierarchy*: still countable, but unwieldy.
- Consequence: if one can ‘interpret’ $\mathcal{N}$ in another structure $\mathcal{N}'$, then $\mathcal{N}'$ is just as complicated.
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Model theory: Definitions

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- **Order-minimal, or o-minimal model** $\mathcal{M}$: All definable subsets of $M^1$ are finite unions of intervals (convex sets with both infema and suprema inside $M$). (Slang: A formula $\varphi$ only “changes its mind” finitely many times on $\mathcal{M}$, with definable changing points.)
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- **Weakly o-minimal model** \( \mathcal{M} \): All definable subsets of \( M^1 \) are finite unions of convex sets (not necessarily intervals). (Slang: A formula \( \varphi \) only “changes its mind” finitely many times on \( \mathcal{M} \).)
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- May discuss a strongly minimal theory, such that every model of the theory is also strongly minimal.
In any infinite linear order without endpoints, strong minimality is not possible (as witnessed by the formula $x > a$ for any $a$). Hence, whenever a linear ordering is present in the model, o-minimal is the simplest possible case.

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- Example: $\mathbb{Q}$ considered as a divisible ordered abelian group, $(\mathbb{Q}, +, <, 0)$, is o-minimal (also via quantifier elimination)...
o-minimal models and theories

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- Example: $\mathbb{Q}$ considered as a divisible ordered abelian group, $(\mathbb{Q}, +, <, 0)$, is o-minimal (also via quantifier elimination)...
- ... but, $(\mathbb{Q}, +, \cdot, <, 0, 1)$, the ordered field, is *not* o-minimal. In fact, this is the *worst* possible case, as one can define the integers in this structure with a single formula (Robinson 1949).
The main tool in the study of o-minimal structures is the Monotonicity Theorem (Pillay, Steinhorn 1986):

[Consider all linearly ordered models to be densely ordered.] Suppose \( \mathcal{M} \) is o-minimal, and let \( f : M \to M \) be a definable function. Then there is a definable finite partition of \( \text{dom} \, f \) into intervals \( \{U_i : i \leq n\} \) such that for each \( i \), \( f|U_i \) is strictly monotone and continuous.
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**Cellular decomposition:** (Knight, Pillay, Steinhorn 1986) The idea: Given an o-minimal structure $\mathcal{M}$, every definable subset of $M^n$ can be broken down into finitely many *cells*, where a cell can be thought of as a “nice” subset of $M^n$. 
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[Diagram of a face with shaded regions representing cells.]
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o-minimal: main results

**Theorem:** (Knight, Pillay, Steinhorn 1986) Let $\mathcal{M}$ be o-minimal, and $\mathcal{N} \equiv \mathcal{M}$. Then $\mathcal{N}$ is o-minimal. (Uses cell decomposition): “Every o-minimal model has an o-minimal theory.”

From this fact, the study of o-minimal *models* becomes the study of o-minimal *theories*. Once you know a theory is o-minimal, *cellular decomposition* shows that any such structure is quite manageable in terms of definable sets.
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- Big Question: Which structures have o-minimal theories?

- Subquestion: $(\mathbb{R}, +, \cdot, <, 0, 1)$, the real field, is o-minimal. Which expansions (by language) of the real field are o-minimal?
Example: Take \((\mathbb{Q}, <)\), the rational line, and add a unary predicate symbol \(P\) to the language. Let \(\mathcal{M}\) be \((\mathbb{Q}, <, P)\), where \(P^\mathcal{M}\) is interpreted as \(\{q \in \mathbb{Q} : -\pi < q < \pi\}\), or \((-\pi, \pi) \cap \mathbb{Q}\).
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The structure seems quite tame - but \(P\) does not define an interval: there is no supremum or infimum in \(\mathbb{Q}\)... hence the structure is \textit{not} o-minimal.
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By a theorem of Poizat, this structure is weakly o-minimal.
Weakly o-minimal structures (II)

Three avenues of study:

- What is true in a weakly o-minimal structure? In particular, which tameness properties are preserved from o-minimal structures?
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- Which structures are weakly o-minimal?
- Given an o-minimal structure, which of its expansions by language are weakly o-minimal?
Weakly o-minimal structures: Failures

Investigate the possible analogues of o-minimal theories to weakly o-minimal structures:

- Monotonicity fails.
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- Monotonicity fails.
- Thus, cell decomposition fails.
- There exist $\mathcal{M} \equiv \mathcal{N}$ with $\mathcal{M}$ weakly o-minimal, and $\mathcal{N}$ not weakly o-minimal. Hence not every weakly o-minimal structure has a weakly o-minimal theory.
Weak monotonicity

A failure of monotonicity:
Let $\mathcal{M} = (\mathbb{Q}_L + (\mathbb{Q} \times \mathbb{Q})_R, <, f)$, where $<$ mimics the lexicographic order on $(\mathbb{Q} \times \mathbb{Q})_R$, and $f : (\mathbb{Q} \times \mathbb{Q})_R \rightarrow \mathbb{Q}_L$ such that $f((p_R, q_R)) = p_L$. 
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Note that in this case, $f$ is still \textit{locally} constant everywhere, but not globally constant.
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Theorem: Let \(\mathcal{M}\) be weakly o-minimal, and \(f: M \to M\) be a definable function. Then there is a definable finite partition of \(\text{dom}(f)\) into convex sets \(U_i\) such that for each \(i\), \(f|U_i\) is continuous and either locally strictly increasing, locally strictly decreasing, or locally constant.
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- Weak monotonicity result leads to a version of cell decomposition without continuous boundary functions.
- Still not very useful in establishing tameness of definable sets in higher dimensions.
Weak monotonicity (III)

Definition: A weakly o-minimal structure $\mathcal{M}$ has strong monotonicity iff for every definable function $f : M \rightarrow M$, there is a definable finite partition of $\text{dom}(f)$ into convex sets $U_i$ such that for each $i$, $f|U_i$ is strictly monotone and continuous.
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Big question: When do we have strong monotonicity?
A New Approach

Stability

- Stable theories have particularly nice geometries and have simple definable sets.

Simplicity

- More recently, simple theories (a subclass of stable theories) have been studied and have similar geometric properties.
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Is there a parallel concept for ordered structures?
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- The concept is still being tested.
Definition: A cut of a linearly ordered structure $\mathcal{M}$ is a pair of subsets $(C, D)$ such that $C, D$ are convex, $C < D$ and $C \cup D = M$. 

If $\mathcal{M}$ is a weakly o-minimal expansion of an ordered field with no valuational cut, then $\mathcal{M}$ satisfies strong monotonicity. (MacPherson, Marker, Steinhorn 2000) In the above case, an analogue of cellular decomposition holds as well. Note: the weaker version of cell decomposition does not guarantee a weakly o-minimal theory.
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Note: the weaker version of cell decomposition does not guarantee a weakly o-minimal theory.
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- Simpler question: look at structures with weakly o-minimal theories expanding the theory of ordered groups.
- Ask: is there a slight weakening of strong monotonicity that will allow locally constant functions, but still achieve strong monotonicity of decreasing and increasing functions?
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- Goal: find a set of conditions that guarantee strong monotonicity.
- Simpler question: look at structures with weakly o-minimal theories expanding the theory of ordered groups.
- Ask: is there a slight weakening of strong monotonicity that will allow locally constant functions, but still achieve strong monotonicity of decreasing and increasing functions?
- Does rosiness guarantee strong monotonicity?