Definable choice for a class of weakly o-minimal structures

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joint with Chris Laskowski

1. Big picture
   - Weak o-minimality
   - Valuational vs. nonvaluational
2. Good / Bad / Ugly
   - Bad
   - Good
   - Ugly
3. Nonvaluational structures
   - van den Dries test
   - A failure
4. Valuational structures
   - T-convexity
   - T-resistance

O-minimal structures

A (densely ordered) structure $(\mathcal{M}, <, \ldots)$ is *o-minimal* if every definable subset (with parameters) of $\mathcal{M}$ is a finite union of **intervals.** (**Intervals** have endpoints in the universe of $\mathcal{M}$.)

Examples:

- DLO (although we'll assume all structures expand a group)
- DOAG
- RCOF [Tarski-Seidenberg, 1930]
- RCOF with exponentiation [Wilkie, 1996]

Prominent non-example: $(\mathbb{Q}, +, \cdot, <, 0, 1)$, in which $\mathbb{N}$ is definable. [Robinson, 1949]

Weakly o-minimal structures

A (densely ordered) structure $(\mathcal{N}, <, \ldots)$ is *weakly o-minimal* if every definable subset (with parameters) of $\mathcal{N}$ is a finite union of **convex sets**. (**Convex sets** don't necessarily have endpoints in the universe of $\mathcal{N}$.)

Examples:

- Any o-minimal structure
- RCVF [Cherlin-Dickmann, 1983]
- Any o-minimal structure after adding a predicate for a new convex set [Baisalov-Poizat, 1998]
- Any weakly o-minimal structure after adding a collection of new unary predicates for convex sets [Baizhanov, 2001]

Big picture questions

- What does it mean for an ordered structure to be “nice”?
- Which of these properties are true for o-minimal structures?
- Under what conditions do these translate to weakly o-minimal structures?

O-minimality is nice

- O-minimality is determined by the theory [Knight-Pillay-Steinhorn, 1986].
- O-minimal structures have prime models over sets [Knight-Pillay-Steinhorn, 1986].
- Definable subsets of o-minimal structures obey a monotonicity property, and are subject to a cellular decomposition [Pillay-Steinhorn, 1984 & Knight-Pillay-Steinhorn, 1986].
- An o-minimal structure has definable choice (possibly after adding constants) [van den Dries, 1998].

Weakly o-minimal?

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Valuational & nonvaluational structures

[Macpherson-Marker-Steinhorn, 2000] A weakly o-minimal structure expanding a group $\mathcal{M}$ is **valuational** if there is a convex definable subset U which is bounded above and ε from M such that $U + \varepsilon = U$. If not, the structure is said to be **nonvaluational**.

Nonvaluational

Let $\mathcal{M} = (\mathbb{Q}, +, <, P)$, where $P^{\mathcal{M}} = \{x \in \mathbb{Q} : x < \pi\}$.

Valuational

Let $\mathcal{N} = (\mathbb{R}^*, +, <, U)$, where $\mathbb{R}^*$ is an end extension of $\mathbb{R}$, and $U^{\mathcal{N}}$ is the set of elements bounded above by a standard real.

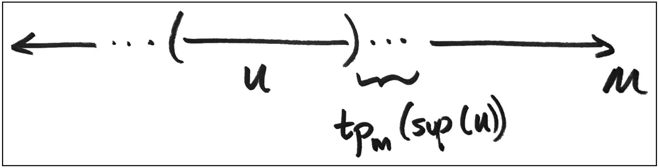
Note: $\mathcal{M}$ is valuational if and only if it has a definable nontrivial proper subgroup – for a fixed U, let G_U be the set of ε which satisfy the definition above.

Notation and convention

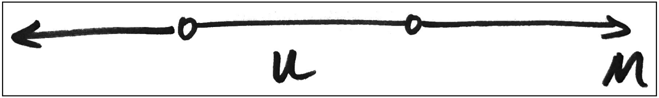
- If $\mathcal{M}$ is o-minimal and U is a new convex subset, then $(\mathcal{M}, U)$ is weakly o-minimal and has a weakly o-minimal theory T.
- A *cut* of M is a partition of M into nonempty disjoint convex sets C and D such that $C < D$. The *cut determined by U* is composed of the downward closure of U on the left, and the set of elements of M which are greater than U on the right.
- In this case we will say interchangeably that $\mathcal{M}$ is (non)valuational, $(\mathcal{M}, U)$ is (non)valuational, and U or the cut determined by U is (non)valuational.
- Let $tp_{\mathcal{M}} sup(U)$ be the type "x is greater than every element of U and x is smaller than all elements of M which are greater than U." We call this the *type of the cut determined by U.*

Topological approach

If U is valuational the realizations of $tp_{\mathcal{M}} sup(U)$, in any extension which realizes it, necessarily occupy a great deal of space, enough to fit an entire coset of the subgroup G_U:

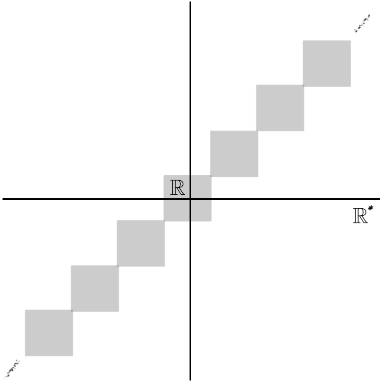


If U is nonvaluational, it does not:



Cells in valuational structures

There is no hope of a perfectly analogous cellular decomposition. In $(\mathbb{R}^*, +, <, U)$, define $X \subseteq (\mathbb{R}^*)^2$ by the formula $U(y - x)$.



Similarly, the general monotonicity theorem for weakly o-minimal structures only guarantees *local* monotonicity.

Nonvaluational results

[McPherson-Marker-Steinhorn, 2000]
If $\mathcal{F}$ is a weakly o-minimal nonvaluational *field*:

- $\mathcal{F}$ is real closed;
- $\mathcal{F}$ satisfies *strong monotonicity*;
- $\mathcal{F}$ has uniform finiteness and a *strong cellular decomposition*;
- $\mathcal{F}$ has a weakly o-minimal theory.

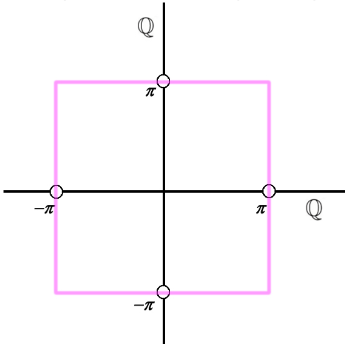
[Wencel, 2010]
If $\mathcal{G}$ is a weakly o-minimal nonvaluational *group*:

- $\mathcal{G}$ satisfies *strong* monotonicity;
- $\mathcal{G}$ has uniform finiteness and a *strong cellular decomposition*;
- $\mathcal{G}$ has a weakly o-minimal theory.

“Strong”

Strong cellular decomposition

The definition of *cell* is necessarily more complex than in the o-minimal case. Consider $(\mathbb{Q}, +, <, P)$ where $P = (-\pi, \pi)$.



Cell defined by $P(x) \wedge P(y)$

Strong cells in nonvaluational structures

- Cellular boundary functions: $f : \mathcal{M} \to \overline{\mathcal{M}}$, for some reasonable notion of $\overline{\mathcal{M}}$, a completion of $\mathcal{M}$ with respect to definable cuts.
- Insist boundary functions be strictly monotone and *strongly continuous*, meaning that they *look like* continuous functions (as opposed to locally continuous).
- Adjust the monotonicity theorem similarly.
- A nonvaluational weakly o-minimal group is "almost o-minimal", in the following sense [Wencel, 2008]:
    - $(\mathcal{M}, +, <, \ldots)$ can be extended canonically to an o-minimal structure $(\overline{\mathcal{M}}, +, <, \overline{\mathcal{C}})$ where $\overline{\mathcal{C}}$ is the set of completions of $\mathcal{M}$-cells.
    - For every definable subset $X \subseteq \overline{M}$, $X \cap M$ is definable in $\mathcal{M}$.

Definable choice

Definition

T has *definable choice* if, for any $\mathcal{M} \models T$ and any $\varphi(\overline{x}, y)$, there is a definable unary F such that

1. If $\overline{a} \in M$ and $\mathcal{M} \models \exists y \varphi(\overline{a}, y)$, then $\mathcal{M} \models \varphi(\overline{a}, F(\overline{a}))$, and
2. If $\{b \in M : \mathcal{M} \models \varphi(\overline{a}, b)\} = \{b \in M : \mathcal{M} \models \varphi(\overline{a}', b)\}$, then $F(\overline{a}) = F(\overline{a}')$.

The first condition states that F is a *definable Skolem function* for φ. The second specifies that F in fact chooses a *unique* representative of each class $\varphi(\overline{a}, x)$.

Definable choice in a valuational structure

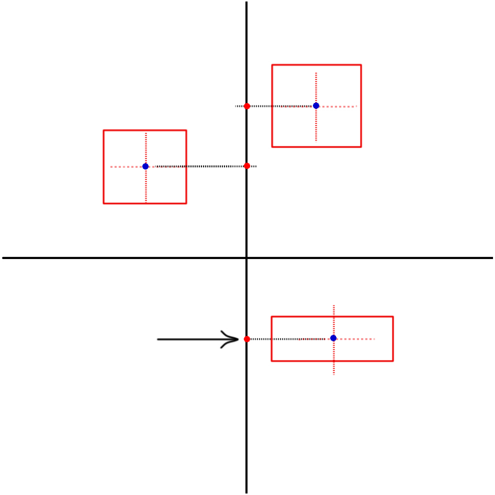
Lemma [S, 2008, maybe others?]: If $\mathcal{M}$ is a valuational weakly o-minimal structure which expands a group, then $\mathcal{M}$ does not satisfy definable choice.

Proof: Let G be a definable convex proper subgroup of $\mathcal{M}$, and $\varphi(x, y) = G(x - y)$. Then φ defines an equivalence relation with infinitely many convex classes of nonzero width; a set of representatives would be a definable infinite discrete subset of $\mathcal{M}$.

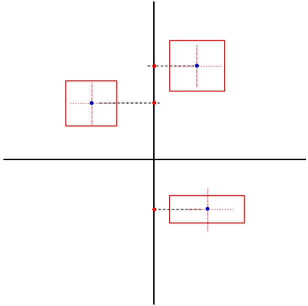
What about the nonvaluational case?

Skolem functions

Our focus will primarily be on Skolem functions. Any o-minimal structure $\mathcal{M}$ expanding a group has definable Skolem functions after naming a positive element. Visually:



Skolem functions



- In a weakly o-minimal structure, we have definable sets without endpoints in the structure.
- In turn, it can be harder to name interior points of intervals.

A note about adding constants

- Say a model $\mathcal{M}$ with theory T is *Skolem-definable* is every $\overline{a}$-definable subset of M^n contains an $\overline{a}$-definable element.
- Obviously having definable Skolem functions is strictly stronger than being Skolem-definable.
- Without having some constant symbols in the language, even o-minimal structures may fail to be Skolem definable (think $x > 0$ in $(\mathbb{Q}, +, <)$).
- However, in the o-minimal case, from any open interval one can locate its endpoints and midpoint or, if infinite, add or subtract a positive element to find an interior point.
- With non-interval convex sets, this is not true; for this reason we may have to, at minimum, add nonzero constants for elements inside and outside our convex sets.

van den Dries test

Theorem [van den Dries 1984]

If T admits q.e., then TFAE:

- T has definable Skolem functions.
- Each model $\mathcal{M} \models T_\forall$ has an extension $\overline{\mathcal{M}} \models T$ which is algebraic over $\mathcal{M}$ and rigid over $\mathcal{M}$.

- "U is a valuational subset" is a $\forall$-sentence, provided we add a constant symbol for a "small" element ε realizing the definition.
- But "U is a nonvaluational subset" is an $\forall\exists$-sentence, which means models of the universal theory of $\mathcal{M}$ may not be nonvaluational.

Example

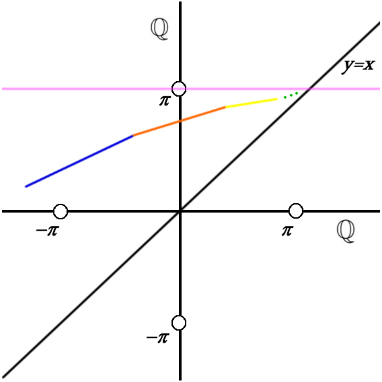
- Let $\mathcal{M}$ be the Hahn sum $(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}, +, <)$, (where $<^{\mathcal{M}}$ is lexicographic and $+^{\mathcal{M}}$ is componentwise), and define $\mathcal{N} = (\mathcal{M}, U)$ where $U^{\mathcal{N}} = \{\bar{x} \in M : \bar{x} < (1, 1, \pi)\}$.
- Weakly o-minimal since $(\mathbb{Q}^3, +, <) \models DOAG$ and $U^{\mathcal{N}}$ is convex.
- Nonvaluational since every 'small' element is some $(0, 0, q)$, and $\mathbb{Q}$ is Archimedean.
- $\mathcal{N}'$ is the substructure generated by $+^{\mathcal{N}}$ with universe $\{(n, p, q) : p = 0\}$... which is valuational, witnessed by $(0, 0, 1)$.
- $\mathcal{N}''$ is the substructure with universe $\{(n, p, q) : q = 0\}$. Then $U^{\mathcal{N}''}$ is the interval $(-\infty, (1, 1, 0))$... so $\mathcal{M}''$ is o-minimal.

Example: failure of Skolem functions by exhaustion

Let $\mathcal{N} = (\mathbb{Q}, +, <, P, \lambda_q)_{q\in\mathbb{Q}}$: ($\mathbb{Q}$ as a $\mathbb{Q}$-vector space, together with a predicate for $-\pi < x < \pi$).

$\mathcal{N}$ has q.e., so we can say what the definable functions are.

- Every definable function is piecewise linear, with a rational slope.
- Let $\varphi(x, y) = P(x) \wedge P(y) \wedge x < y$.
- A Skolem function for φ is a function $F : \mathbb{Q} \to \mathbb{Q}$ such that $x < f(x) < \pi$.



Generalizing the nonvaluational result

Theorem [S, 2008]

$(\mathcal{M}, +, <, \ldots)$ o-minimal, and *U* a new nonvaluational left-closed convex subset. Then $(\mathcal{M}, U)$ does not have definable Skolem functions after naming constants (and hence does not satisfy definable choice).

Proof (Outline):

- Let $b \in \mathfrak{C}$ realize $tp_{\mathcal{M}}(supU)$.
- $(\mathcal{M}, U)$ is nonvaluational if and only if this type is a uniquely realizable irrational cut (in the sense of Marker [1986]).
- Aside: A nonuniquely realizable cut occurs if *U* is valuational, and a noncut (or "rational type") occurs only if *U* is definable in $\mathcal{M}$ to begin with.

Generalizing the nonvaluational result

- O-minimal structures have prime models over sets; let $\mathcal{N} = pr(\mathcal{M} \cup \{b\})$.
- *Lemma:* $\mathcal{M}$ is dense in $\mathcal{N}$, if and only if $tp_{\mathcal{M}}(b)$ is uniquely realized in $\mathcal{N}$, if and only if U is nonvaluational.
- *Theorem* [van den Dries, 1998]: If $\mathcal{M}$ is dense in $\mathcal{N}$ then every $(\mathcal{N}, \mathcal{M})$-definable function $F : \mathcal{M} \to \mathcal{M}$ is piecewise $\mathcal{M}$-definable.
- If there is a definable $F : \mathcal{M} \to \mathcal{M}$ such that $x < F(x) \in U$ for all x, then $(\mathcal{M}, U) \models \lim_{x \to \sup(U)} F(x) = \sup(U)$.
- By density there is an $(\mathcal{N}, \mathcal{M})$-definable G whose graph restricted to M is equal to F (replace $U(x)$ with $x < b$).
- Being piecewise $\mathcal{M}$-definable means there is an interval in $\mathcal{M}$ on which $G \restriction M$ is continuous and always below b.
- This interval must overlap the cut, meaning "$F(b) < b$."
- Note, this also follows from an argument of Pillay-Steinhorn [1986] about sequential completeness.

O-minimal traces

A recent approach [Eleftheriou-Hasson-Keren, 2016] adds further generality.

Definition

$\mathcal{M}$ is an *o-minimal trace* if there is an o-minimal expansion of a group $\mathcal{N}$ in a language $\mathcal{L}$ such that $M \subsetneq N$ is dense in N, $\mathcal{M} \upharpoonright \mathcal{L} \preceq \mathcal{N}$, and $\mathcal{M}$ is the structure induced on M from $\mathcal{N}$.

Theorem: If $\mathcal{M}$ is elementarily equivalent to a reduct of an o-minimal trace preserving the ordered group structure, then $\mathcal{M}$ does not have definable Skolem functions (even after naming constants).

T-convexity

Fix T, an o-minimal expansion of a ring; let $\mathcal{M} \models T$. A subset $V \subsetneq M$ is *T-convex* if V is convex, and for any 0-definable continuous total function $F : \mathcal{M} \to \mathcal{M}$, $F(V) \subseteq V$. T_{convex} is the theory of such a pair $(\mathcal{M}, V)$... (which happens to be valuational weakly o-minimal).

Theorem [van den Dries-Lewenberg, 1995]

If T eliminates quantifiers and is universally axiomatizable, then T_{convex} eliminates quantifiers and is complete.

Corollary [van den Dries, 1997]

Add a new constant c to the language; interpret c by an element outside of V. Then $T_{convex}(c)$ has definable Skolem functions.

T-resistance

T-convexity was originally used in an analysis of RCVF; for more generality, we expand the definition to include expansion of a group.

Fix T, an o-minimal expansion of a group; let $\mathcal{M} \models T$. $V \subsetneq M$ is *T-resistant* if V is convex and the language of $\mathcal{M}$ contains a constant for an element of V, and for any 0-definable continuous total function $F : \mathcal{M} \to \mathcal{M}$, $F(V) \subseteq V$. T_{res} is the theory of the pair $(\mathcal{M}, V)$.

If $\mathcal{M}$ is a field, then $(\mathcal{M}, V)$ is T-resistant iff it is T-convex.

Theorem [L-S, 2016]

If T eliminates quantifiers and is universally axiomatizable, then T_{res} eliminates quantifiers and is complete.

Corollary [L-S, 2016]

Add a new constant $c_{\notin V}$. Then $T_{res}(c_{\notin V})$ has definable Skolem functions.

Extending the power of T-resistance

Main theorem [L-S, 2016]

Let $\mathcal{M}$ be an o-minimal expansion of a group with a definable positive element, and $U \subseteq M$ be a symmetric convex valuational subset. If $\mathrm{dcl}_{\mathcal{M}}(\emptyset) \subseteq U$, then $(\mathcal{M}, U)$ is T-resistant.

As our main tool for the proof of the main theorem, we define *pluslike functions*, which capture the generalized topological properties of addition.

- Given $\mathcal{M}$ expanding an ordered group, call a definable $F : M^2 \to M$ is *pluslike* if it is continuous on M, and strictly increasing in both variables.
- ($+_{\mathcal{M}}$ is of course pluslike.)
- For F pluslike on $\mathcal{M}$ and a definable downward-closed convex subset U of $\mathcal{M}$, we say that U is F-*valuational* if there is $\varepsilon > 0$ from M such that for all $a \in C$, $F(a, \varepsilon) \in C$.

Pluslike functions

Proposition

Let $\mathcal{M}$ be o-minimal, and U a predicate for a new downward-closed convex subset. TFAE:

1. U is F-valuational for some definable pluslike function F.
2. U is F-valuational for all definable pluslike functions F.
3. U is valuational.
4. $tp_{\mathcal{M}} sup(U)$ is nonuniquely realizable.

The meat of the proof is (1)⇒(4)⇒(2), and relies on the monotonicity of the function in order to construct a partial inverse of F for values around the edge of the cut.

Proving the main theorem

If $(\mathcal{M}, U)$ is not T-resistant, by using the monotonicity on $\mathcal{M}$, we can find a 0-definable (in $\mathcal{M}$) continuous total strictly increasing function $G : M \to M$, and $\alpha < \beta \in U$ such that $G(\alpha) \in U$ and $G(\beta) > U$.

Given this function G, the function $F(x, y) := G(x + y)$ is definable and pluslike; yet, the cut represented by U is not F-valuational, since for any $\varepsilon > 0$, $F(\beta, \varepsilon) > G(\beta) > U$. $\dashv$

The final piece of this puzzle for us is that if $\mathcal{M}$ is o-minimal and U is convex and valuational, we may look at elementary extensions to get a model of $Th(\mathcal{M}, U)$ in which $\mathrm{dcl}_{\mathcal{M}}(\emptyset) \in U$.

Corollary: If $\mathcal{M}$ is an o-minimal expansion of a group and U is a new convex valuational subset, then $(\mathcal{M}, U)$ has definable Skolem functions *after naming constants*.

A few words about computing Skolem functions

We have a direct quantifier elimination and complete algorithm for Skolem functions in the case of a structure $(\mathcal{M}, U)$ which is *T-immune* (strictly stronger than T-resistant). [S-2008]

At its heart: given a definable F, how do you definably choose a value y so that $\models U(F(\overline{a}, y))$?

Use regular cellular decomposition to assume $F_{\overline{a}}$ is continuous and monotone (say, increasing) on an interval (α, β). For a fixed small ε, provided $F(\overline{a}, \alpha) \in U$, we know $F(\overline{a}, \alpha + \varepsilon)$ is guaranteed to be in U as well.

This sort of maneuver *doesn't* work when U is nonvaluational.

Open questions

- [Eleftheriou-Hasson-Keren, 2016] conjecture that no nonvaluational structures have definable Skolem functions.
- Are there nonvaluational structures that do have external limits?
- In a weakly o-minimal valuational structure that does have external limits, can we still have definable Skolem functions after adding constants?
- Given an o-minimal trace obtained by adding a single new convex set, is there a single function that can be added to Skolemize the theory?

Thanks and further reading

- van den Dries, Tame Topology and O-minimal Structures, London Math. Soc. Lecture Note Series #248 (survey), 1998
- van den Dries & Lewenberg, *T-convexity and tame extensions*, JSL 60(1), 1995
- Macpherson, Marker, & Steinhorn, *Weakly o-minimal structures and real-closed fields*, Trans. AMS 352(12), 2000