Definable choice for a class of weakly o-minimal structures

Christopher Shaw
Columbia College Chicago

November 15, 2016

joint with Chris Laskowski
1. Big picture
   - Weak o-minimality
   - Valuational vs. nonvaluational

2. Good / Bad / Ugly
   - Bad
   - Good
   - Ugly

3. Nonvaluational structures
   - van den Dries test
   - A failure

4. Valuational structures
   - $T$-convexity
   - $T$-resistance
A (densely ordered) structure \((\mathcal{M}, <, \ldots)\) is \textit{o-minimal} if every definable subset (with parameters) of \(\mathcal{M}\) is a finite union of \textit{intervals}. (\textit{Intervals} have endpoints in the universe of \(\mathcal{M}\).)

Examples:

- DLO (although we’ll assume all structures expand a group)
- DOAG
- RCOF [Tarski-Seidenberg, 1930]
- RCOF with exponentiation [Wilkie, 1996]

Prominent non-example: \((\mathbb{Q}, +, \cdot, <, 0, 1)\), in which \(\mathbb{N}\) is definable. [Robinson, 1949]
Weakly o-minimal structures

A (densely ordered) structure $(\mathcal{N}, <, \ldots)$ is *weakly o-minimal* if every definable subset (with parameters) of $\mathcal{N}$ is a finite union of convex sets. (Convex sets don’t necessarily have endpoints in the universe of $\mathcal{N}$.)

Examples:

- Any o-minimal structure
- RCVF [Cherlin-Dickmann, 1983]
- Any o-minimal structure after adding a predicate for a new convex set [Baisalov-Poizat, 1998]
- Any weakly o-minimal structure after adding a collection of new unary predicates for convex sets [Baizhanov, 2001]
What does it mean for an ordered structure to be “nice”?

Which of these properties are true for o-minimal structures?

Under what conditions do these translate to weakly o-minimal structures?
O-minimality is nice

- O-minimality is determined by the theory [Knight-Pillay-Steinhorn, 1986].
- O-minimal structures have prime models over sets [Knight-Pillay-Steinhorn, 1986].
- Definable subsets of o-minimal structures obey a monotonicity property, and are subject to a cellular decomposition [Pillay-Steinhorn, 1984 & Knight-Pillay-Steinhorn, 1986].
- An o-minimal structure has definable choice (possibly after adding constants) [van den Dries, 1998].
Weakly o-minimal?

- O-minimality is determined by the theory [Knight-Pillay-Steinhorn, 1986].
- O-minimal structures have prime models over sets [Knight-Pillay-Steinhorn, 1986].
- Definable subsets of o-minimal structures obey a monotonicity property, and are subject to a cellular decomposition [Pillay-Steinhorn, 1984 & Knight-Pillay-Steinhorn, 1986].
- An o-minimal structure has definable choice (possibly after adding constants) [van den Dries, 1998].
Valuational & nonvaluational structures

[Macpherson-Marker-Steinhorn, 2000] A weakly o-minimal structure expanding a group \( \mathcal{M} \) is \textbf{valuational} if there is a convex definable subset \( U \) which is bounded above and \( \varepsilon \) from \( M \) such that \( U + \varepsilon = U \). If not, the structure is said to be \textbf{nonvaluational}.

### Nonvaluational

Let \( \mathcal{M} = (\mathbb{Q}, +, <, P) \), where \( P^\mathcal{M} = \{ x \in \mathbb{Q} : x < \pi \} \).

### Valuational

Let \( \mathcal{N} = (\mathbb{R}^*, +, <, U) \), where \( \mathbb{R}^* \) is an end extension of \( \mathbb{R} \), and \( U^\mathcal{N} \) is the set of elements bounded above by a standard real.

Note: \( \mathcal{M} \) is valuational if and only if it has a definable nontrivial proper subgroup – for a fixed \( U \), let \( G_U \) be the set of \( \varepsilon \) which satisfy the definition above.
Notation and convention

- If $\mathcal{M}$ is o-minimal and $U$ is a new convex subset, then $(\mathcal{M}, U)$ is weakly o-minimal and has a weakly o-minimal theory $T$.

- A cut of $M$ is a partition of $M$ into nonempty disjoint convex sets $C$ and $D$ such that $C < D$. The cut determined by $U$ is composed of the downward closure of $U$ on the left, and the set of elements of $M$ which are greater than $U$ on the right.

- In this case we will say interchangeably that $\mathcal{M}$ is (non)valuational, $(\mathcal{M}, U)$ is (non)valuational, and $U$ or the cut determined by $U$ is (non)valuational.

- Let $tp_{{\mathcal{M}}} sup(U)$ be the type “$x$ is greater than every element of $U$ and $x$ is smaller than all elements of $M$ which are greater than $U$.” We call this the type of the cut determined by $U$. 

Shaw

Big picture: Valuational vs. nonvaluational

11/15/16
If $U$ is valuational, the realizations of $tp_{M} sup(U)$, in any extension which realizes it, necessarily occupy a great deal of space, enough to fit an entire coset of the subgroup $G_{U}$:

If $U$ is nonvaluational, it does not:
There is no hope of a perfectly analogous cellular decomposition. In \((\mathbb{R}^*, +, <, U)\), define \(X \subseteq (\mathbb{R}^*)^2\) by the formula \(U(y - x)\).

Similarly, the general monotonicity theorem for weakly o-minimal structures only guarantees \textit{local} monotonicity.
Nonvaluational results

[McPherson-Marker-Steinhorn, 2000]
If $\mathcal{F}$ is a weakly o-minimal nonvaluational field:

- $\mathcal{F}$ is real closed;
- $\mathcal{F}$ satisfies strong monotonicity;
- $\mathcal{F}$ has uniform finiteness and a strong cellular decomposition;
- $\mathcal{F}$ has a weakly o-minimal theory.

[Wencel, 2010]
If $\mathcal{G}$ is a weakly o-minimal nonvaluational group:

- $\mathcal{G}$ satisfies strong monotonicity;
- $\mathcal{G}$ has uniform finiteness and a strong cellular decomposition;
- $\mathcal{G}$ has a weakly o-minimal theory.
Strong cellular decomposition
The definition of \textit{cell} is necessarily more complex than in the o-minimal case. Consider \((\mathbb{Q}, +, <, P)\) where \(P = (-\pi, \pi)\).

Cell defined by \(P(x) \land P(y)\)
Cellular boundary functions: \( f : \mathcal{M} \to \overline{\mathcal{M}} \), for some reasonable notion of \( \overline{\mathcal{M}} \), a completion of \( \mathcal{M} \) with respect to definable cuts.

Insist boundary functions be strictly monotone and *strongly continuous*, meaning that they *look like* continuous functions (as opposed to locally continuous).

Adjust the monotonicity theorem similarly.

A nonvaluational weakly o-minimal group is “almost o-minimal”, in the following sense [Wencel, 2008]:

- \((\mathcal{M}, +, <, \ldots)\) can be extended canonically to an o-minimal structure \((\overline{\mathcal{M}}, +, <, \overline{C})\) where \(\overline{C}\) is the set of completions of \(\mathcal{M}\)-cells.
- For every definable subset \(X \subseteq \overline{M}\), \(X \cap M\) is definable in \(\mathcal{M}\).
Definable choice

Definition

*T* has *definable choice* if, for any *M* ⊨ *T* and any *φ(\bar{x}, y)*, there is a definable unary *F* such that

1. If \( \bar{a} \in M \) and \( M \models \exists y \varphi(\bar{a}, y) \), then \( M \models \varphi(\bar{a}, F(\bar{a})) \), and
2. If \( \{ b \in M : M \models \varphi(\bar{a}, b) \} = \{ b \in M : M \models \varphi(\bar{a}', b) \} \), then \( F(\bar{a}) = F(\bar{a}') \).

The first condition states that *F* is a *definable Skolem function* for *φ*. The second specifies that *F* in fact chooses a *unique* representative of each class *φ(\bar{a}, x)*.
**Lemma** [S, 2008, maybe others?]: If $\mathcal{M}$ is a valuational weakly o-minimal structure which expands a group, then $\mathcal{M}$ does not satisfy definable choice.

**Proof:** Let $G$ be a definable convex proper subgroup of $\mathcal{M}$, and $\varphi(x, y) = G(x - y)$. Then $\varphi$ defines an equivalence relation with infinitely many convex classes of nonzero width; a set of representatives would be a definable infinite discrete subset of $\mathcal{M}$.

What about the nonvaluational case?
Skolem functions

Our focus will primarily be on Skolem functions. Any o-minimal structure $\mathcal{M}$ expanding a group has definable Skolem functions after naming a positive element. Visually:
In a weakly o-minimal structure, we have definable sets without endpoints in the structure.

In turn, it can be harder to name interior points of intervals.
A note about adding constants

- Say a model $\mathcal{M}$ with theory $T$ is **Skolem-definable** is every $\bar{a}$-definable subset of $M^n$ contains an $\bar{a}$-definable element.
- Obviously having definable Skolem functions is strictly stronger than being Skolem-definable.
- Without having some constant symbols in the language, even o-minimal structures may fail to be Skolem definable (think $x > 0$ in $(\mathbb{Q}, +, <)$).
- However, in the o-minimal case, from any open interval one can locate its endpoints and midpoint or, if infinite, add or subtract a positive element to find an interior point.
- With non-interval convex sets, this is not true; for this reason we may have to, at minimum, add nonzero constants for elements inside and outside our convex sets.
Theorem [van den Dries 1984]
If $T$ admits q.e., then TFAE:
- $T$ has definable Skolem functions.
- Each model $\mathcal{M} \models T$ has an extension $\bar{\mathcal{M}} \models T$ which is algebraic over $\mathcal{M}$ and rigid over $\mathcal{M}$.

- “$U$ is a valuational subset” is a $\forall$-sentence, provided we add a constant symbol for a “small” element $\varepsilon$ realizing the definition.
- But “$U$ is a nonvaluational subset” is an $\forall \exists$-sentence, which means models of the universal theory of $\mathcal{M}$ may not be nonvaluational.
Example

- Let $\mathcal{M}$ be the Hahn sum $(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}, +, <)$, (where $<^\mathcal{M}$ is lexicographic and $+^\mathcal{M}$ is componentwise), and define $\mathcal{N} = (\mathcal{M}, U)$ where $U^\mathcal{N} = \{ \bar{x} \in M : \bar{x} < (1, 1, \pi) \}$.
- Weakly o-minimal since $(\mathbb{Q}^3, +, <) \models DOAG$ and $U^\mathcal{N}$ is convex.
- Nonvaluational since every ‘small’ element is some $(0, 0, q)$, and $\mathbb{Q}$ is Archimedean.
- $\mathcal{N}'$ is the substructure generated by $+^\mathcal{N}$ with universe $\{ (n, p, q) : p = 0 \}$...which is valuational, witnessed by $(0, 0, 1)$.
- $\mathcal{N}''$ is the substructure with universe $\{ (n, p, q) : q = 0 \}$. Then $U^{\mathcal{N}''}$ is the interval $(-\infty, (1, 1, 0))$...so $\mathcal{M}''$ is o-minimal.
Example: failure of Skolem functions by exhaustion

Let $\mathcal{N} = (\mathbb{Q}, +, <, P, \lambda q)_{q \in \mathbb{Q}}$: (\mathbb{Q} as a \mathbb{Q}-vector space, together with a predicate for $-\pi < x < \pi$).

$\mathcal{N}$ has q.e., so we can say what the definable functions are.

- Every definable function is piecewise linear, with a rational slope.
- Let $\varphi(x, y) = P(x) \land P(y) \land x < y$.
- A Skolem function for $\varphi$ is a function $F : \mathbb{Q} \to \mathbb{Q}$ such that $x < f(x) < \pi$. 
Generalizing the nonvaluational result

**Theorem [S, 2008]**

$(\mathcal{M}, +, <, \ldots)$ o-minimal, and $U$ a new nonvaluational left-closed convex subset. Then $(\mathcal{M}, U)$ does not have definable Skolem functions after naming constants (and hence does not satisfy definable choice).

**Proof** (Outline):

- Let $b \in \mathcal{C}$ realize $tp_\mathcal{M}(\text{sup}U)$.
- $(\mathcal{M}, U)$ is nonvaluational if and only if this type is a uniquely realizable irrational cut (in the sense of Marker [1986]).
- Aside: A nonuniquely realizable cut occurs if $U$ is valuational, and a noncut (or “rational type”) occurs only if $U$ is definable in $\mathcal{M}$ to begin with.
Generalizing the nonvaluational result

- O-minimal structures have prime models over sets; let $\mathcal{N} = \text{pr} (\mathcal{M} \cup \{ b \})$.

**Lemma:** $\mathcal{M}$ is dense in $\mathcal{N}$, if and only if $tp_{\mathcal{M}}(b)$ is uniquely realized in $\mathcal{N}$, if and only if $U$ is nonvaluational.

**Theorem** [van den Dries, 1998]: If $\mathcal{M}$ is dense in $\mathcal{N}$ then every $(\mathcal{N}, \mathcal{M})$-definable function $F : \mathcal{M} \rightarrow \mathcal{M}$ is piecewise $\mathcal{M}$-definable.

If there is a definable $F : \mathcal{M} \rightarrow \mathcal{M}$ such that $x < F(x) \in U$ for all $x$, then $(\mathcal{M}, U) \models \lim_{x \to \sup(U)} F(x) = \sup(U)$.

By density there is an $(\mathcal{N}, \mathcal{M})$-definable $G$ whose graph restricted to $M$ is equal to $F$ (replace $U(x)$ with $x < b$).

Being piecewise $\mathcal{M}$-definable means there is an interval in $\mathcal{M}$ on which $G \upharpoonright M$ is continuous and always below $b$.

This interval must overlap the cut, meaning “$F(b) < b$.”

Note, this also follows from an argument of Pillay-Steinhorn [1986] about sequential completeness.

**Definition**

$\mathcal{M}$ is an *o-minimal trace* if there is an o-minimal expansion of a group $\mathcal{N}$ in a language $\mathcal{L}$ such that $\mathcal{M} \subsetneq \mathcal{N}$ is dense in $\mathcal{N}$, $\mathcal{M} \upharpoonright \mathcal{L} \preceq \mathcal{N}$, and $\mathcal{M}$ is the structure induced on $\mathcal{M}$ from $\mathcal{N}$.

**Theorem:** If $\mathcal{M}$ is elementarily equivalent to a reduct of an o-minimal trace preserving the ordered group structure, then $\mathcal{M}$ does not have definable Skolem functions (even after naming constants).
Fix $T$, an o-minimal expansion of a ring; let $\mathcal{M} \models T$. A subset $V \subsetneq \mathcal{M}$ is $T$-convex if $V$ is convex, and for any 0-definable continuous total function $F : \mathcal{M} \to \mathcal{M}$, $F(V) \subseteq V$. $T_{\text{convex}}$ is the theory of such a pair $(\mathcal{M}, V)$... (which happens to be valuational weakly o-minimal).

**Theorem [van den Dries-Lewenberg, 1995]**

If $T$ eliminates quantifiers and is universally axiomatizable, then $T_{\text{convex}}$ eliminates quantifiers and is complete.

**Corollary [van den Dries, 1997]**

Add a new constant $c$ to the language; interpret $c$ by an element outside of $V$. Then $T_{\text{convex}}(c)$ has definable Skolem functions.
$T$-resistance

$T$-convexity was originally used in an analysis of RCVF; for more generality, we expand the definition to include expansion of a group.

Fix $T$, an o-minimal expansion of a group; let $\mathcal{M} \models T$. $V \subsetneq M$ is $T$-resistant if $V$ is convex and the language of $\mathcal{M}$ contains a constant for an element of $V$, and for any 0-definable continuous total function $F : \mathcal{M} \to \mathcal{M}$, $F(V) \subseteq V$. $T_{\text{res}}$ is the theory of the pair $(\mathcal{M}, V)$.

If $\mathcal{M}$ is a field, then $(\mathcal{M}, V)$ is $T$-resistant iff it is $T$-convex.

**Theorem [L-S, 2016]**

If $T$ eliminates quantifiers and is universally axiomatizable, then $T_{\text{res}}$ eliminates quantifiers and is complete.

**Corollary [L-S, 2016]**

Add a new constant $c \notin V$. Then $T_{\text{res}}(c \notin V)$ has definable Skolem functions.
Extending the power of $T$-resistance

Main theorem [L-S, 2016]

Let $\mathcal{M}$ be an o-minimal expansion of a group with a definable positive element, and $U \subseteq M$ be a symmetric convex valuational subset. If $\operatorname{dcl}_\mathcal{M}(\emptyset) \subseteq U$, then $(\mathcal{M}, U)$ is $T$-resistant.

As our main tool for the proof of the main theorem, we define *pluslike functions*, which capture the generalized topological properties of addition.

- **Given** $\mathcal{M}$ expanding an ordered group, call a definable $F : M^2 \to M$ is *pluslike* if it is continuous on $M$, and strictly increasing in both variables.
- $(+_\mathcal{M}$ is of course pluslike.)
- For $F$ pluslike on $\mathcal{M}$ and a definable downward-closed convex subset $U$ of $\mathcal{M}$, we say that $U$ is $F$-valuational if there is $\varepsilon > 0$ from $M$ such that for all $a \in C$, $F(a, \varepsilon) \in C$.  

Shaw
Valuational structures: $T$-resistance
Proposition

Let $\mathcal{M}$ be o-minimal, and $U$ a predicate for a new downward-closed convex subset. TFAE:

1. $U$ is $F$-valuational for some definable pluslike function $F$.
2. $U$ is $F$-valuational for all definable pluslike functions $F$.
3. $U$ is valuational.
4. $\text{tp}_{\mathcal{M}} \sup(U)$ is nonuniquely realizable.

The meat of the proof is (1)$\Rightarrow$(4)$\Rightarrow$(2), and relies on the monotonicity of the function in order to construct a partial inverse of $F$ for values around the edge of the cut.
If \((\mathcal{M}, U)\) is not \(T\)-resistant, by using the monotonicity on \(\mathcal{M}\), we can find a 0-definable (in \(\mathcal{M}\)) continuous total strictly increasing function \(G : M \to M\), and \(\alpha < \beta \in U\) such that \(G(\alpha) \in U\) and \(G(\beta) > U\).

Given this function \(G\), the function \(F(x, y) := G(x + y)\) is definable and pluslike; yet, the cut represented by \(U\) is not \(F\)-valuational, since for any \(\varepsilon > 0\), \(F(\beta, \varepsilon) > G(\beta) > U\).

The final piece of this puzzle for us is that if \(\mathcal{M}\) is o-minimal and \(U\) is convex and valuational, we may look at elementary extensions to get a model of \(Th(\mathcal{M}, U)\) in which \(dcl_{\mathcal{M}}(\emptyset) \in U\).

**Corollary:** If \(\mathcal{M}\) is an o-minimal expansion of a group and \(U\) is a new convex valuational subset, then \((\mathcal{M}, U)\) has definable Skolem functions after naming constants.
We have a direct quantifier elimination and complete algorithm for Skolem functions in the case of a structure \((\mathcal{M}, U)\) which is \(T\)-immune (strictly stronger than \(T\)-resistant). [S-2008]

At its heart: given a definable \(F\), how do you definably choose a value \(y\) so that \(\models U(F(\bar{a}, y))\)?

Use regular cellular decomposition to assume \(F_{\bar{a}}\) is continuous and monotone (say, increasing) on an interval \((\alpha, \beta)\). For a fixed small \(\varepsilon\), provided \(F(\bar{a}, \alpha) \in U\), we know \(F(\bar{a}, \alpha + \varepsilon)\) is guaranteed to be in \(U\) as well.

This sort of maneuver \textit{doesn’t} work when \(U\) is nonvaluational.
Open questions

- [Eleftheriou-Hasson-Keren, 2016] conjecture that no nonvaluational structures have definable Skolem functions.
- Are there nonvaluational structures that do have external limits?
- In a weakly o-minimal valuational structure that does have external limits, can we still have definable Skolem functions after adding constants?
- Given an o-minimal trace obtained by adding a single new convex set, is there a single function that can be added to Skolemize the theory?

van den Dries & Lewenberg, *T-convexity and tame extensions*, JSL 60(1), 1995

Macpherson, Marker, & Steinhorn, *Weakly o-minimal structures and real-closed fields*, Trans. AMS 352(12), 2000