Definable choice for a class of weakly o-minimal structures

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ASL Winter Meeting 2011

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Background

- O-minimality
- Weakly o-minimal structures
- Definable subgroups
- Pathologies

Nonvaluational structures

- van den Dries test
- Failure of Skolem functions

Valuational structures

- $T$-convexity

Future work
1. **Background**
   - O-minimality
     - Weakly o-minimal structures
     - Definable subgroups
     - Pathologies

2. **Nonvaluational structures**
   - van den Dries test
   - Failure of Skolem functions

3. **Valuational structures**
   - $T$-convexity

4. **Future work**
A (densely ordered) structure \((M, <, \ldots)\) is *o-minimal* if every definable subset (with parameters) of \(M^1\) is a finite union of intervals.
A (densely ordered) structure \((\mathcal{M}, <, \ldots)\) is \textit{o-minimal} if every definable subset (with parameters) of \(\mathcal{M}^1\) is a finite union of intervals.

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- An o-minimal structure has an o-minimal theory [Knight-Pillay-Steinhorn, 1986].
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- “Definable choice” [van den Dries, 1998]
Skolem functions (2)

Example

[Diagram of four quadrants with red boxes indicating examples]
Skolem functions (2)

Example
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Example

[Diagram showing a cross-section with points and lines]
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- Any o-minimal structure is weakly o-minimal.
- Any weakly o-minimal structure which is Dedekind complete is also o-minimal.
- Thus, any weakly o-minimal structure with universe \(\mathbb{R}\) is o-minimal.
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$\mathcal{M}_1 = (\mathbb{Q}, <, +, P)$, where $P^{\mathcal{M}_1} = \{x \in \mathbb{Q} : -\pi < x < \pi\}$.
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$\mathcal{M}_2 = (\mathbb{R}^*, <, +, \cdot, U)$, where $\mathbb{R}^*$ is a proper end extension of $\mathbb{R}$, and $U^{\mathcal{M}_2}$ is the convex hull of $\mathbb{R}$ in $\mathbb{R}^*$. 
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- $\mathcal{M}_1$ and $\mathcal{M}_2$: two main paradigms we explore in this talk.
Theorem [Poizat, 1998]

Let $\mathcal{M} = (M, <, \ldots)$ be o-minimal, and let $\mathcal{U}$ be a set of unary predicate symbols, and $L' = L \cup \mathcal{U}$. Let $\mathcal{M}'$ be the expanded $L'$-structure $(M, <, \mathcal{U}, \ldots)$, with each $U \in \mathcal{U}$ interpreted by a convex set. Then $\mathcal{M}'$ is weakly o-minimal.
A note on obtaining weakly o-minimal structures

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- Baizhanov (2001) proved this in more generality, allowing $\mathcal{M}$ itself to be weakly o-minimal.
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- Monotonicity?
- Cellular decomposition?
- Definable choice?
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Chris Laskowski & Christopher Shaw
Definable choice in weakly o-minimal structures
An o-minimal group has no proper definable subgroup.
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  - Weakly o-minimal, but $U(\mathcal{M})$ defines a proper subgroup.
- Any definable subgroup is convex $\Rightarrow$ weakly o-minimal groups satisfy DOAG.
Valuational: definition and examples

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A weakly o-minimal group $\mathcal{M}$ is valuational if $\mathcal{M}$ has a definable valuational cut; equivalently, $\mathcal{M}$ has a definable proper subgroup.
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![Diagram showing a cut $\langle C, D \rangle$ in $\mathcal{M}$ with points labeled 0, $\varepsilon$, $g$, $g+\varepsilon$, and $D$.]
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- By the subgroup characterization, the proper end-extension of $\mathbb{R}$ is valuational.
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There is no hope of a tame cellular decomposition. In \((\mathbb{R}^*,+,<,U)\), define \(X \subseteq (\mathbb{R}^*)^2\) by the formula \(U(y-x)\).
Cells in valuational structures

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Skolem functions

In a weakly o-minimal structure, we have definable sets without endpoints in the structure. Harder to name midpoints.

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[McPherson-Marker-Steinhorn, 2000]
For a weakly o-minimal nonvaluational group $G$:

- $G$ has a strong monotonicity;
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Definable choice in weakly o-minimal structures
Theorem [van den Dries 1984]

If $T$ admits q.e., then TFAE:

- $T$ has definable Skolem functions.
- Each model $M \models T$ has an extension $M' \models T$ which is algebraic over $M$ and rigid over $M$.

Troublesome for the nonvaluational case: "$M$ is nonvaluational" is an $\exists \forall$-sentence.
Theorem [van den Dries 1984]

If $T$ admits q.e., then TFAE:

1. $T$ has definable Skolem functions.
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If $T$ admits q.e., then TFAE:

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Troublesome for the nonvaluational case: “$\mathcal{M}$ is nonvaluational” is an $\exists \forall$-sentence.
Taking on water

\[ M = (\mathbb{Q}^3, +, <, U), \text{ where } <^M \text{ is lexicographic, } +^M \text{ is componentwise addition in } \mathbb{Q}, \text{ and } U^M = \{ \bar{x} \in \mathbb{Q}^3 : \bar{x} < (1, 1, \pi) \} = \{ (n, p, q) : n \leq 1, p \leq 1, q < \pi \}. \]
Taking on water

- $\mathcal{M} = (\mathbb{Q}^3, +, <, U)$, where $<^\mathcal{M}$ is lexicographic, $+^\mathcal{M}$ is componentwise addition in $\mathbb{Q}$, and $U^\mathcal{M} = \{\bar{x} \in \mathbb{Q}^3 : \bar{x} < (1, 1, \pi)\} = \{(n, p, q) : n \leq 1, p \leq 1, q < \pi\}$.

- Weakly o-minimal since $(\mathbb{Q}^3, +, <) \models DOAG$ and $U^\mathcal{M}$ is convex.
Taking on water

- $\mathcal{M} = (\mathbb{Q}^3, +, <, U)$, where $<_\mathcal{M}$ is lexicographic, $+_\mathcal{M}$ is component-wise addition in $\mathbb{Q}$, and $U^{\mathcal{M}} = \{ \bar{x} \in \mathbb{Q}^3 : \bar{x} < (1, 1, \pi) \} = \{ (n, p, q) : n \leq 1, p \leq 1, q < \pi \}$.
- Weakly o-minimal since $(\mathbb{Q}^3, +, <) \models DOAG$ and $U^{\mathcal{M}}$ is convex.
- Nonvaluational since every ‘small’ element is some $(0, 0, q)$, and $\{q_1\} \times \{q_2\} \times \mathbb{Q}$ is nonvaluational.
\[ M = (\mathbb{Q}^3, +, <, U), \text{ where } <^M \text{ is lexicographic, } +^M \text{ is componentwise addition in } \mathbb{Q}, \text{ and } U^M = \{ \bar{x} \in \mathbb{Q}^3 : \bar{x} < (1, 1, \pi) \} = \{ (n, p, q) : n \leq 1, p \leq 1, q < \pi \}. \]

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\(M'\) is the substructure generated by \(+^M\) with universe \(\{(n, p, q) : p = 0\}...\)
Taking on water

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- $\mathcal{M}'$ is the substructure generated by $+^\mathcal{M}$ with universe $\{(n, p, q) : p = 0\}$... which is valuational.
\( \mathcal{M} = (\mathbb{Q}^3, +, <, U) \), where \(<^\mathcal{M}\) is lexicographic, \(+^\mathcal{M}\) is componentwise addition in \(\mathbb{Q}\), and \(U^\mathcal{M} = \{ \bar{x} \in \mathbb{Q}^3 : \bar{x} < (1, 1, \pi) \} = \{(n, p, q) : n \leq 1, p \leq 1, q < \pi \} \).

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\(\mathcal{M}'\) is the substructure generated by \(+^\mathcal{M}\) with universe \(\{(n, p, q) : p = 0\}\)... which is valuational.

\(\mathcal{M}''\) is the substructure with universe \(\{(n, p, q) : q = 0\}\). Then \(U^\mathcal{M}''\) is an interval...
\[ \mathcal{M} = (\mathbb{Q}^3, +, <, U), \text{ where } <^\mathcal{M} \text{ is lexicographic, } +^\mathcal{M} \text{ is componentwise addition in } \mathbb{Q}, \text{ and } U^\mathcal{M} = \{ \bar{x} \in \mathbb{Q}^3 : \bar{x} < (1, 1, \pi) \} = \{ (n, p, q) : n \leq 1, p \leq 1, q < \pi \}. \]

Weakly o-minimal since \((\mathbb{Q}^3, +, <) \models DOAG\) and \(U^\mathcal{M}\) is convex.

Nonvaluational since every ‘small’ element is some \((0, 0, q)\), and \(\{q_1\} \times \{q_2\} \times \mathbb{Q}\) is nonvaluational.

\(\mathcal{M}'\) is the substructure generated by \(+^\mathcal{M}\) with universe \(\{(n, p, q) : p = 0\}\) which is valuational.

\(\mathcal{M}''\) is the substructure with universe \(\{(n, p, q) : q = 0\}\). Then \(U^\mathcal{M}''\) is an interval... so \(\mathcal{M}''\) is o-minimal.
Background

- O-minimality
- Weakly o-minimal structures
- Definable subgroups
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Nonvaluational structures

- van den Dries test
- Failure of Skolem functions

Valuational structures

- \(T\)-convexity

Future work

Chris Laskowski & Christopher Shaw
Definable choice in weakly o-minimal structures
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Generalizing the nonvaluational result

Theorem [L-S, 2008]

\((\mathcal{M}, +, <, \ldots)\) o-minimal, and \(U\) a new nonvaluational left-closed convex subset. Then \((\mathcal{M}, U)\) does not have definable Skolem functions.
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- **Lemma**: $\mathcal{M}$ is dense in $\mathcal{N}$ if and only if $U$ is nonvaluational.
- **Theorem** [van den Dries, 1998]: Every $(\mathcal{N}, \mathcal{M})$-definable function $F : \mathcal{M} \rightarrow \mathcal{M}$ is piecewise $\mathcal{M}$-definable.
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3 Valuational structures
   - $T$-convexity

4 Future work
Definition of $T$-convexity

Fix $T$, an o-minimal expansion of a ring; let $\mathcal{M} \models T$. A subset $V \subset M$ is $T$-convex if $V$ is convex, and for any 0-definable continuous total function $F : \mathcal{M} \to \mathcal{M}$, $F(V) \subseteq V$. 

$T$-convex is the theory of such a pair $(\mathcal{M}, V)$. This is valuational weakly o-minimal.

Theorem [van den Dries-Lewenberg, 1995]
If $T$ eliminates quantifiers and is universally axiomatizable, then $T$-convex eliminates quantifiers and is complete.

Corollary [van den Dries, 1997]
Add a new constant $c$ to the language; interpret $c$ by an element outside of $V$. Then $T$-convex, $c$ has definable Skolem functions.
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A sense in which valuational structures are $T$-convex

For $V \subseteq M$ a convex subset, say that a 0-definable function $G : M \to M$ is $V$-fast if there is $\varepsilon \in V$ with $G(\varepsilon) \in V$ and $a \in V$ such that $G(a) \notin V$. 
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$V$-fast functions are natural and intuitive witnesses to the failure of $T$-convexity for the structure $(M, V)$. 

Theorem [L-S, 2010]: Let $M$ be an o-minimal expansion of a ring, and $V \subseteq M$ be properly convex such that $\text{acl}(\emptyset) \subseteq V$. If there is a $V$-fast 0-definable continuous total function $G : M \to M$, then $V$ defines a nonvaluational cut. 

Proof uses Marker's analysis of types in o-minimal structures (1986).
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- If \((\mathcal{M}, U)\) is nonvaluational, it does not have definable Skolem functions.

- If \((\mathcal{M}, U)\) is valuational, \((\mathcal{M}, U)\) has a convex definable subgroup, thus cannot have Skolem functions and elimination of imaginaries. Therefore if \((\mathcal{M}, U)\) has Skolem functions and elimination of imaginaries, \(U\) must be an interval, and thus \((\mathcal{M}, U)\) is o-minimal.

- If \((\mathcal{M}, U)\) is \(T\)-convex, then \((\mathcal{M}, U)\) has Skolem functions, but still has a definable proper subgroup, thus \((\mathcal{M}, U)\) does not eliminate imaginaries.
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- We have an explicit calculation of Skolem functions for ‘$T$-immune’ theories (strictly stronger than $T$-convex). Expand this argument to find a direct calculation for a general $T$-convex theory.
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